

# THE TRIANGULATED CATEGORY OF $K$ -MOTIVES $DK_-^{eff}(k)$

GRIGORY GARKUSHA AND IVAN PANIN

ABSTRACT. For any perfect field  $k$  a triangulated category of  $K$ -motives  $DK_-^{eff}(k)$  is constructed in the style of Voevodsky's construction of the category  $DM_-^{eff}(k)$ . To each smooth  $k$ -variety  $X$  the  $K$ -motive  $M_{\mathbb{K}}(X)$  is associated in the category  $DK_-^{eff}(k)$  and

$$K_n(X) = \mathrm{Hom}_{DK_-^{eff}(k)}(M_{\mathbb{K}}(X)[n], M_{\mathbb{K}}(pt)), \quad n \in \mathbb{Z},$$

where  $pt = \mathrm{Spec}(k)$  and  $K(X)$  is Quillen's  $K$ -theory of  $X$ .

## 1. INTRODUCTION

The Voevodsky triangulated category of motives  $DM_-^{eff}(k)$  [15] provides a natural framework to study motivic cohomology. In [1] the authors constructed a triangulated category of  $K$ -motives providing a natural framework for such a fundamental object as the motivic spectral sequence. The main idea was to use a kind of motivic algebra of spectral categories and modules over them.

In this paper an alternative approach to constructing a triangulated category of  $K$ -motives is presented. We work in the framework of strict  $V$ -spectral categories introduced in the paper (Definition 2.5). The main feature of such a spectral category  $\mathcal{O}$  is that it is connective and Nisnevich excisive in the sense of [1] and  $\pi_0\mathcal{O}$ -(pre)sheaves, where  $\pi_0\mathcal{O}$  is a ringoid associated to  $\mathcal{O}$ , share lots of common properties with (pre)sheaves with transfers (or  $Cor$ -(pre)sheaves) in the sense of Voevodsky [14].

To any strict  $V$ -spectral category over  $k$ -smooth varieties we associate a triangulated category  $D\mathcal{O}_-^{eff}(k)$ , which in spirit is constructed similar to  $DM_-^{eff}(k)$  (Section 3). For instance, the ringoid of correspondences  $Cor$  gives rise to a strict  $V$ -spectral category  $\mathcal{O} = \mathcal{O}_{cor}$  whenever the base field  $k$  is perfect. In this case the Voevodsky category  $DM_-^{eff}(k)$  is recovered as the category  $D\mathcal{O}_-^{eff}(k)$  (Corollary 3.6).

The main  $V$ -spectral category  $\mathbb{K}$  is constructed in Section 4. It is strict over perfect fields. The associated triangulated category  $D\mathcal{O}_-^{eff}(k)$  is denoted by  $DK_-^{eff}(k)$ . The spectral category  $\mathbb{K}$  is a priori different from spectral categories constructed by the authors in [1] and has some advantages over spectral categories of [1].

To each smooth  $k$ -variety  $X$  we associate its  $K$ -motive  $M_{\mathbb{K}}(X)$ . By definition, it is an object of the category  $DK_-^{eff}(k)$ . We prove in Theorem 5.11 that

$$K_n(X) = \mathrm{Hom}_{DK_-^{eff}(k)}(M_{\mathbb{K}}(X)[n], M_{\mathbb{K}}(pt)), \quad n \in \mathbb{Z},$$

where  $pt = \mathrm{Spec}(k)$  and  $K(X)$  is Quillen's  $K$ -theory of  $X$ . Thus Quillen's  $K$ -theory is represented by the  $K$ -motive of the point.

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The spectral category  $\mathbb{K}$  is of great utility in authors' paper [2], in which they solve some problems related to the motivic spectral sequence. In fact, the problems were the main motivation for constructing the spectral category  $\mathbb{K}$  and developing the machinery of  $K$ -motives.

Throughout the paper we denote by  $Sm/k$  the category of smooth separated schemes of finite type over the base field  $k$ .

## 2. PRELIMINARIES

We work in the framework of spectral categories and modules over them in the sense of Schwede–Shipley [11]. We start with preparations.

Recall that symmetric spectra have two sorts of homotopy groups which we shall refer to as *naive* and *true homotopy groups* respectively following terminology of [10]. Precisely, the  $k$ th naive homotopy group of a symmetric spectrum  $X$  is defined as the colimit

$$\hat{\pi}_k(X) = \operatorname{colim}_n \pi_{k+n} X_n.$$

Denote by  $\gamma X$  a stably fibrant model of  $X$  in  $Sp^\Sigma$ . The  $k$ -th true homotopy group of  $X$  is given by

$$\pi_k X = \hat{\pi}_k(\gamma X),$$

the naive homotopy groups of the symmetric spectrum  $\gamma X$ .

Naive and true homotopy groups of  $X$  can considerably be different in general (see, e.g., [5, 10]). The true homotopy groups detect stable equivalences, and are thus more important than the naive homotopy groups. There is an important class of *semistable* symmetric spectra within which  $\hat{\pi}_*$ -isomorphisms coincide with  $\pi_*$ -isomorphisms. Recall that a symmetric spectrum is semistable if some (hence any) stably fibrant replacement is a  $\pi_*$ -isomorphism. Suspension spectra, Eilenberg–Mac Lane spectra,  $\Omega$ -spectra or  $\Omega$ -spectra from some point  $X_n$  on are examples of semistable symmetric spectra (see [10]). So Waldhausen's algebraic  $K$ -theory symmetric spectrum we shall use later is semistable. Semistability is preserved under suspension, loop, wedges and shift.

A symmetric spectrum  $X$  is *n-connected* if the true homotopy groups of  $X$  are trivial for  $k \geq n$ . The spectrum  $X$  is *connective* if it is  $(-1)$ -connected, i.e., its true homotopy groups vanish in negative dimensions.  $X$  is *bounded below* if  $\pi_i(X) = 0$  for  $i \leq 0$ .

**Definition 2.1.** (1) Following [11] a *spectral category* is a category  $\mathcal{O}$  which is enriched over the category  $Sp^\Sigma$  of symmetric spectra (with respect to smash product, i.e., the monoidal closed structure of [5, 2.2.10]). In other words, for every pair of objects  $o, o' \in \mathcal{O}$  there is a morphism symmetric spectrum  $\mathcal{O}(o, o')$ , for every object  $o$  of  $\mathcal{O}$  there is a map from the sphere spectrum  $S$  to  $\mathcal{O}(o, o)$  (the “identity element” of  $o$ ), and for each triple of objects there is an associative and unital composition map of symmetric spectra  $\mathcal{O}(o', o'') \wedge \mathcal{O}(o, o') \rightarrow \mathcal{O}(o, o'')$ . An  $\mathcal{O}$ -module  $M$  is a contravariant spectral functor to the category  $Sp^\Sigma$  of symmetric spectra, i.e., a symmetric spectrum  $M(o)$  for each object of  $\mathcal{O}$  together with coherently associative and unital maps of symmetric spectra  $M(o) \wedge \mathcal{O}(o', o) \rightarrow M(o')$  for pairs of objects  $o, o' \in \mathcal{O}$ . A morphism of  $\mathcal{O}$ -modules  $M \rightarrow N$  consists of maps of symmetric spectra  $M(o) \rightarrow N(o)$  strictly compatible with the action of  $\mathcal{O}$ . The category of  $\mathcal{O}$ -modules will be denoted by  $\operatorname{Mod} \mathcal{O}$ .

(2) A *spectral functor* or a *spectral homomorphism*  $F$  from a spectral category  $\mathcal{O}$  to a spectral category  $\mathcal{O}'$  is an assignment from  $\operatorname{Ob} \mathcal{O}$  to  $\operatorname{Ob} \mathcal{O}'$  together with morphisms  $\mathcal{O}(a, b) \rightarrow \mathcal{O}'(F(a), F(b))$  in  $Sp^\Sigma$  which preserve composition and identities.

(3) The *monoidal product*  $\mathcal{O} \wedge \mathcal{O}'$  of two spectral categories  $\mathcal{O}$  and  $\mathcal{O}'$  is the spectral category where  $\operatorname{Ob}(\mathcal{O} \wedge \mathcal{O}') := \operatorname{Ob} \mathcal{O} \times \operatorname{Ob} \mathcal{O}'$  and  $\mathcal{O} \wedge \mathcal{O}'((a, x), (b, y)) := \mathcal{O}(a, b) \wedge \mathcal{O}'(x, y)$ .

(4) A spectral category  $\mathcal{O}$  is said to be *connective* if for any objects  $a, b$  of  $\mathcal{O}$  the spectrum  $\mathcal{O}(a, b)$  is connective.

(5) By a ringoid over  $Sm/k$  we mean a preadditive category  $\mathcal{R}$  whose objects are those of  $Sm/k$  together with a functor

$$\rho : Sm/k \rightarrow \mathcal{R},$$

which is identity on objects. Every such ringoid gives rise to a spectral category  $\mathcal{O}_{\mathcal{R}}$  whose objects are those of  $Sm/k$  and the morphisms spectrum  $\mathcal{O}_{\mathcal{R}}(X, Y)$ ,  $X, Y \in Sm/k$ , is the Eilenberg–Mac Lane spectrum  $H\mathcal{R}(X, Y)$  associated with the abelian group  $\mathcal{R}(X, Y)$ . Given a map of schemes  $\alpha$ , its image  $\rho(\alpha)$  will also be denoted by  $\alpha$ , dropping  $\rho$  from notation.

(6) By a spectral category over  $Sm/k$  we mean a spectral category  $\mathcal{O}$  whose objects are those of  $Sm/k$  together with a spectral functor

$$\sigma : \mathcal{O}_{naive} \rightarrow \mathcal{O},$$

which is identity on objects. Here  $\mathcal{O}_{naive}$  stands for the spectral category whose morphism spectra are defined as

$$\mathcal{O}_{naive}(X, Y)_p = \text{Hom}_{Sm/k}(X, Y)_+ \wedge S^p$$

for all  $p \geq 0$  and  $X, Y \in Sm/k$ .

It is straightforward to verify that the category of  $\mathcal{O}_{naive}$ -modules can be regarded as the category of presheaves  $Pre^{\Sigma}(Sm/k)$  of symmetric spectra on  $Sm/k$ . This is used in the sequel without further comment.

Let  $\mathcal{O}$  be a spectral category and let  $\text{Mod } \mathcal{O}$  be the category of  $\mathcal{O}$ -modules. Recall that the projective stable model structure on  $\text{Mod } \mathcal{O}$  is defined as follows (see [11]). The weak equivalences are the objectwise stable weak equivalences and fibrations are the objectwise stable projective fibrations. The stable projective cofibrations are defined by the left lifting property with respect to all stable projective acyclic fibrations.

Recall that the Nisnevich topology is generated by elementary distinguished squares, i.e. pullback squares

$$(1) \quad \begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & \mathcal{Q} & \downarrow \varphi \\ U & \xrightarrow{\psi} & X \end{array}$$

where  $\varphi$  is etale,  $\psi$  is an open embedding and  $\varphi^{-1}(X \setminus U) \rightarrow (X \setminus U)$  is an isomorphism of schemes (with the reduced structure). Let  $\mathcal{Q}$  denote the set of elementary distinguished squares in  $Sm/k$  and let  $\mathcal{O}$  be a spectral category over  $Sm/k$ . By  $\mathcal{Q}_{\mathcal{O}}$  denote the set of squares

$$(2) \quad \begin{array}{ccc} \mathcal{O}(-, U') & \longrightarrow & \mathcal{O}(-, X') \\ \downarrow & \mathcal{O}\mathcal{Q} & \downarrow \varphi \\ \mathcal{O}(-, U) & \xrightarrow{\psi} & \mathcal{O}(-, X) \end{array}$$

which are obtained from the squares in  $\mathcal{Q}$  by taking  $X \in Sm/k$  to  $\mathcal{O}(-, X)$ . The arrow  $\mathcal{O}(-, U') \rightarrow \mathcal{O}(-, X')$  can be factored as a cofibration  $\mathcal{O}(-, U') \rightarrow \text{Cyl}$  followed by a simplicial homotopy equivalence  $\text{Cyl} \rightarrow \mathcal{O}(-, X')$ . There is a canonical morphism  $A_{\mathcal{O}\mathcal{Q}} := \mathcal{O}(-, U) \sqcup_{\mathcal{O}(-, U')} \text{Cyl} \rightarrow \mathcal{O}(-, X)$ .

**Definition 2.2** (see [1]). I. The *Nisnevich local model structure* on  $\text{Mod } \mathcal{O}$  is the Bousfield localization of the stable projective model structure with respect to the family of projective cofibrations

$$\mathcal{N}_{\mathcal{O}} = \{\text{cyl}(A_{\mathcal{O}Q} \rightarrow \mathcal{O}(-, X))\}_{\mathcal{Q}_{\mathcal{O}}}.$$

The homotopy category for the Nisnevich local model structure will be denoted by  $SH_{S^1}^{\text{nis}} \mathcal{O}$ . In particular, if  $\mathcal{O} = \mathcal{O}_{\text{naive}}$  then we have the Nisnevich local model structure on  $Pre^{\Sigma}(Sm/k) = \text{Mod } \mathcal{O}_{\text{naive}}$  and we shall write  $SH_{S^1}^{\text{nis}}(k)$  to denote  $SH_{S^1}^{\text{nis}} \mathcal{O}_{\text{naive}}$ .

II. The *motivic model structure* on  $\text{Mod } \mathcal{O}$  is the Bousfield localization of the Nisnevich local model structure with respect to the family of projective cofibrations

$$\mathcal{A}_{\mathcal{O}} = \{\text{cyl}(\mathcal{O}(-, X \times \mathbb{A}^1) \rightarrow \mathcal{O}(-, X))\}_{X \in Sm/k}.$$

The homotopy category for the motivic model structure will be denoted by  $SH_{S^1}^{\text{mot}} \mathcal{O}$ . In particular, if  $\mathcal{O} = \mathcal{O}_{\text{naive}}$  then we have the motivic model structure on  $Pre^{\Sigma}(Sm/k) = \text{Mod } \mathcal{O}_{\text{naive}}$  and we shall write  $SH_{S^1}^{\text{mot}}(k)$  to denote  $SH_{S^1}^{\text{mot}} \mathcal{O}_{\text{naive}}$ .

**Definition 2.3** (see [1]). I. We say that  $\mathcal{O}$  is *Nisnevich excisive* if for every elementary distinguished square  $Q$

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & \mathcal{Q} & \downarrow \phi \\ U & \xrightarrow{\psi} & X \end{array}$$

the square  $\mathcal{O}Q$  (2) is homotopy pushout in the Nisnevich local model structure on  $Pre^{\Sigma}(Sm/k)$ .

II.  $\mathcal{O}$  is *motivically excisive* if:

- (A) for every elementary distinguished square  $Q$  the square  $\mathcal{O}Q$  (2) is homotopy pushout in the motivic model structure on  $Pre^{\Sigma}(Sm/k)$  and
- (B) for every  $X \in Sm/k$  the natural map

$$\mathcal{O}(-, X \times \mathbb{A}^1) \rightarrow \mathcal{O}(-, X)$$

is a weak equivalence in the motivic model structure on  $Pre^{\Sigma}(Sm/k)$ .

Let  $AffSm/k$  be the full subcategory of  $Sm/k$  whose objects are the smooth affine varieties.  $AffSm/k$  gives rise to a spectral  $\mathcal{O}_{Aff}$  whose objects are those of  $AffSm/k$  and morphisms spectra are defined as

$$\mathcal{O}_{Aff}(X, Y) := \text{Hom}_{AffSm/k}(X, Y)_+ \wedge \mathbb{S},$$

where  $\mathbb{S}$  is the sphere spectrum and  $X, Y \in AffSm/k$ .

Recall that a sheaf  $\mathcal{F}$  of abelian groups in the Nisnevich topology on  $Sm/k$  is *strictly  $\mathbb{A}^1$ -invariant* if for any  $X \in Sm/k$ , the canonical morphism

$$H_{\text{nis}}^*(X, \mathcal{F}) \rightarrow H_{\text{nis}}^*(X \times \mathbb{A}^1, \mathcal{F})$$

is an isomorphism.

**Definition 2.4.** Let  $\mathcal{R}$  be a ringoid over  $Sm/k$  together with the structure functor  $\rho : Sm/k \rightarrow \mathcal{R}$ . We say that  $\mathcal{R}$  is a *V-ringoid* (“V” for Voevodsky) if

- (1) for any elementary distinguished square  $Q$  the sequence of Nisnevich sheaves associated to representable presheaves

$$0 \rightarrow \mathcal{R}_{\text{nis}}(-, U') \rightarrow \mathcal{R}_{\text{nis}}(-, U) \oplus \mathcal{R}_{\text{nis}}(-, X') \rightarrow \mathcal{R}_{\text{nis}}(-, X) \rightarrow 0$$

is exact;

(2) there is a functor

$$\boxtimes : \mathcal{R} \times \text{AffSm}/k \rightarrow \mathcal{R}$$

sending  $(X, U) \in \text{Sm}/k \times \text{AffSm}/k$  to  $X \times U \in \text{Sm}/k$  and such that  $1_X \boxtimes \alpha = \rho(1_X \times \alpha)$ ,  $(u + v) \boxtimes \alpha = u \boxtimes \alpha + v \boxtimes \alpha$  for all  $\alpha \in \text{Mor}(\text{AffSm}/k)$  and  $u, v \in \text{Mor}(\mathcal{R})$ .

(3) for any  $\mathcal{R}$ -presheaf of abelian groups  $\mathcal{F}$ , i.e.  $\mathcal{F}$  is a contravariant functor from  $\mathcal{R}$  to abelian groups, the associated Nisnevich sheaf  $\mathcal{F}_{\text{nis}}$  has a unique structure of a  $\mathcal{R}$ -presheaf for which the canonical homomorphism  $\mathcal{F} \rightarrow \mathcal{F}_{\text{nis}}$  is a homomorphism of  $\mathcal{R}$ -presheaves. Moreover, if  $\mathcal{F}$  is homotopy invariant then so is  $\mathcal{F}_{\text{nis}}$ ;

We refer to  $\mathcal{R}$  as a *strict V-ringoid* if every  $\mathbb{A}^1$ -invariant Nisnevich  $\mathcal{R}$ -sheaf is strictly  $\mathbb{A}^1$ -invariant.

We want to make several remarks for the definition. Condition (1) implies the spectral category  $\mathcal{O}_{\mathcal{R}}$  associated to the ringoid  $\mathcal{R}$  is Nisnevich excisive. Condition (2) implies that for any  $\mathcal{R}$ -presheaf  $\mathcal{F}$  and any affine scheme  $U \in \text{AffSm}/k$  the presheaf

$$\underline{\text{Hom}}(U, \mathcal{F}) := \mathcal{F}(- \times U)$$

is an  $\mathcal{R}$ -presheaf. Moreover, it is functorial in  $U$ .

**Definition 2.5.** Let  $\mathcal{O}$  be a spectral category over  $\text{Sm}/k$  together with the structure spectral functor  $\sigma : \mathcal{O}_{\text{naive}} \rightarrow \mathcal{O}$ . We say that  $\mathcal{O}$  is a *V-spectral category* if

- (1)  $\mathcal{O}$  is connective and Nisnevich excisive;
- (2) there is a spectral functor

$$\square : \mathcal{O} \wedge \mathcal{O}_{\text{Aff}} \rightarrow \mathcal{O}$$

sending  $(X, U) \in \text{Sm}/k \times \text{AffSm}/k$  to  $X \times U \in \text{Sm}/k$  and such that  $1_X \square \alpha = \sigma(1_X \times \alpha)$  for all  $\alpha \in \text{Mor}(\text{AffSm}/k)$ ;

- (3)  $\pi_0 \mathcal{O}$  is a V-ringoid such that the structure map  $\rho : \text{Sm}/k \rightarrow \pi_0 \mathcal{O}$  equals the composite map

$$\text{Sm}/k \rightarrow \pi_0 \mathcal{O}_{\text{naive}} \xrightarrow{\pi_0(\sigma)} \pi_0 \mathcal{O}.$$

We also require the structure pairing  $\boxtimes : \pi_0 \mathcal{O} \times \text{AffSm}/k \rightarrow \pi_0 \mathcal{O}$  to be the composite functor

$$\pi_0 \mathcal{O} \times \text{AffSm}/k \rightarrow \pi_0 \mathcal{O} \times \pi_0 \mathcal{O}_{\text{Aff}} \rightarrow \pi_0(\mathcal{O} \wedge \mathcal{O}_{\text{Aff}}) \xrightarrow{\pi_0(\square)} \pi_0 \mathcal{O}.$$

We refer to  $\mathcal{O}$  as a *strict V-spectral category* if the V-ringoid  $\pi_0 \mathcal{O}$  is strict.

We note that if  $\mathcal{O}$  is a V-spectral category, then for every  $\mathcal{O}$ -module  $M$  and any affine smooth scheme  $U$ , the presheaf of symmetric spectra

$$\underline{\text{Hom}}(U, M) := M(- \times U)$$

is an  $\mathcal{O}$ -module. Moreover,  $M(- \times U)$  is functorial in  $U$ .

**Lemma 2.6.** Every V-spectral category  $\mathcal{O}$  is motivically excisive.

*Proof.* Every V-spectral category is, by definition, Nisnevich excisive. Since there is an action of affine smooth schemes on  $\mathcal{O}$ , the fact that  $\mathcal{O}$  is motivically excisive is proved similar to [1, 5.8].  $\square$

Let  $\mathcal{O}$  be a V-spectral category. Since it is both Nisnevich and motivically excisive, it follows from [1, 5.13] that the pair of natural adjoint functors

$$\Psi_* : \text{Pre}^\Sigma(\text{Sm}/k) \rightleftarrows \text{Mod } \mathcal{O} : \Psi^*$$

induces a Quillen pair for the Nisnevich local projective (respectively motivic) model structures on  $\text{Pre}^\Sigma(\text{Sm}/k)$  and  $\text{Mod } \mathcal{O}$ . In particular, one has adjoint functors between triangulated categories

$$(3) \quad \Psi_* : SH_{S^1}^{\text{nis}}(k) \rightleftarrows SH_{S^1}^{\text{nis}} \mathcal{O} : \Psi^* \quad \text{and} \quad \Psi_* : SH_{S^1}^{\text{mot}}(k) \rightleftarrows SH_{S^1}^{\text{mot}} \mathcal{O} : \Psi^*.$$

### 3. THE TRIANGULATED CATEGORY $D\mathcal{O}_-^{eff}(k)$

Throughout this section we work with a strict  $V$ -spectral category  $\mathcal{O}$ . We shall often work with simplicial  $\mathcal{O}$ -modules  $M[\bullet]$ . The *realization* of  $M[\bullet]$  is the  $\mathcal{O}$ -module  $|M|$  defined as the coend

$$|M| = \Delta[\bullet]_+ \wedge_{\Delta} M[\bullet]$$

of the functor  $\Delta[\bullet]_+ \wedge M[\bullet] : \Delta \times \Delta^{op} \rightarrow \text{Mod } \mathcal{O}$ . Here  $\Delta[n]$  is the standard simplicial  $n$ -simplex.

Recall that the simplicial ring  $k[\Delta]$  is defined as

$$k[\Delta]_n = k[x_0, \dots, x_n] / (x_0 + \dots + x_n - 1).$$

By  $\Delta^\cdot$  we denote the cosimplicial affine scheme  $\text{Spec}(k[\Delta])$ . Let

$$M \in \text{Mod } \mathcal{O} \mapsto M_f \in \text{Mod } \mathcal{O}$$

be a fibrant replacement functor in the Nisnevich local model structure on  $\text{Mod } \mathcal{O}$ . Given an  $\mathcal{O}$ -module  $M$ , we set

$$C_*(M) := |\underline{\text{Hom}}(\Delta^\cdot, M_f)|.$$

Note that  $C_*(M)$  is an  $\mathcal{O}$ -module and is functorial in  $M$ . If we regard  $M_f$  as a constant simplicial  $\mathcal{O}$ -module, the map of cosimplicial schemes  $\Delta^\cdot \rightarrow pt$  induces a map of  $\mathcal{O}$ -modules

$$M \rightarrow C_*(M).$$

**Lemma 3.1.** *The functor  $C_*$  respects Nisnevich local weak equivalences. In particular, it induces a triangulated endofunctor*

$$C_* : SH_{S^1}^{\text{nis}} \mathcal{O} \rightarrow SH_{S^1}^{\text{nis}} \mathcal{O}.$$

*Proof.* Let  $\alpha : L \rightarrow M$  be a Nisnevich local weak equivalence of  $\mathcal{O}$ -modules. By [1, 5.12] the forgetful functor  $\Psi^* : \text{Mod } \mathcal{O} \rightarrow \text{Pre}^\Sigma(Sm/k)$  respects Nisnevich local weak equivalences and Nisnevich local fibrant objects. It follows that the fibrant replacement

$$\alpha_f : L_f \rightarrow M_f$$

of  $\alpha$  is a level equivalence of presheaves of ordinary symmetric spectra, and hence so is each map

$$\underline{\text{Hom}}(\Delta^n, \alpha_f) : \underline{\text{Hom}}(\Delta^n, L_f) \rightarrow \underline{\text{Hom}}(\Delta^n, M_f), \quad n \geq 0.$$

Since the realization functor respects level equivalences, our assertion follows.  $\square$

One of advantages of strict  $V$ -spectral categories is that we can construct an  $\mathbb{A}^1$ -local replacement of an  $\mathcal{O}$ -module  $M$  in two steps. We first take  $C_*(M)$  and then its Nisnevich local replacement  $C_*(M)_f$ .

**Theorem 3.2.** *The natural map  $M \rightarrow C_*(M)_f$  is an  $\mathbb{A}^1$ -local replacement of  $M$  in the motivic model structure of  $\mathcal{O}$ -modules.*

*Proof.* The presheaves  $\pi_i(C_*(M))$ ,  $i \in \mathbb{Z}$ , are homotopy invariant and have  $\pi_0 \mathcal{O}$ -transfers. Since  $\mathcal{O}$  is a strict  $V$ -spectral category then each Nisnevich sheaf  $\pi_i^{\text{nis}}(C_*(M)_f)$  is strictly homotopy invariant and has  $\pi_0 \mathcal{O}$ -transfers. By [8, 6.2.7]  $C_*(M)_f$  is  $\mathbb{A}^1$ -local in the motivic model category structure on  $\text{Pre}^\Sigma(Sm/k)$ . By Lemma 2.6  $\mathcal{O}$  is motivically excisive, hence [1, 5.12] implies  $C_*(M)_f$  is  $\mathbb{A}^1$ -local in the motivic model category structure on  $\text{Mod } \mathcal{O}$ .

The map  $M \rightarrow C_*(M)_f$  is the composite

$$M \rightarrow M_f \rightarrow C_*(M) \rightarrow C_*(M)_f.$$

The left and right arrows are Nisnevich local trivial cofibrations. The middle arrow is a level  $\mathbb{A}^1$ -weak equivalence in  $Pre^\Sigma(Sm/k)$  by [9, 3.8]. By Lemma 2.6  $\mathcal{O}$  is motivically excisive, hence [1, 5.12] implies the middle arrow is an  $\mathbb{A}^1$ -weak equivalence in  $Mod \mathcal{O}$ .  $\square$

**Definition 3.3.** The  $\mathcal{O}$ -motive  $M_{\mathcal{O}}(X)$  of a smooth algebraic variety  $X \in Sm/k$  is the  $\mathcal{O}$ -module  $C_*(\mathcal{O}(-, X))$ . We say that an  $\mathcal{O}$ -module  $M$  is *bounded below* if for  $i \ll 0$  the Nisnevich sheaf  $\pi_i^{\text{nis}}(M)$  is zero.  $M$  is *n-connected* if  $\pi_i^{\text{nis}}(M)$  are trivial for  $i \leq n$ .  $M$  is *connective* if it is  $(-1)$ -connected, i.e.,  $\pi_i^{\text{nis}}(M)$  vanish in negative dimensions.

**Corollary 3.4.** *If an  $\mathcal{O}$ -module  $M$  is bounded below (respectively n-connected) then so is  $C_*(M)$ . In particular, the  $\mathcal{O}$ -motive  $M_{\mathcal{O}}(X)$  of any smooth algebraic variety  $X \in Sm/k$  is connective.*

*Proof.* This follows from the preceding theorem and Morel's Connectivity Theorem [8].  $\square$

Denote by  $D\mathcal{O}_-(k)$  the full triangulated subcategory of  $SH_{S^1}^{\text{nis}} \mathcal{O}$  of bounded below  $\mathcal{O}$ -modules. We also denote by  $D\mathcal{O}_-^{\text{eff}}(k)$  the full triangulated subcategory of  $D\mathcal{O}_-(k)$  of those  $\mathcal{O}$ -modules  $M$  such that each Nisnevich sheaf  $\pi_i^{\text{nis}}(M)$  is homotopy invariant.  $D\mathcal{O}_-^{\text{eff}}(k)$  is an analog of Voevodsky's triangulated category  $DM_-^{\text{eff}}(k)$  [15]. We shall show below that  $DM_-^{\text{eff}}(k)$  is equivalent to  $D\mathcal{O}_-^{\text{eff}}(k)$  if  $\mathcal{O} = \mathcal{O}_{\text{cor}}$ .

**Theorem 3.5.** *Let  $\mathcal{O}$  be a strict  $V$ -spectral category. Then the following statements are true:*

(1) *The kernel of  $C_*$  is the full triangulated subcategory  $\mathcal{T}$  of  $SH_{S^1}^{\text{nis}} \mathcal{O}$  generated by the compact objects*

$$\text{cone}(\mathcal{O}(-, X \times \mathbb{A}^1) \rightarrow \mathcal{O}(-, X)), \quad X \in Sm/k.$$

*Moreover,  $C_*$  induces a triangle equivalence of triangulated categories*

$$SH_{S^1}^{\text{nis}} \mathcal{O} / \mathcal{T} \xrightarrow{\sim} SH_{S^1}^{\text{mot}} \mathcal{O}.$$

(2) *The functor*

$$C_* : D\mathcal{O}_-(k) \rightarrow D\mathcal{O}_-(k)$$

*lands in  $D\mathcal{O}_-^{\text{eff}}(k)$ . The kernel of  $C_*$  is  $\mathcal{T}_- := \mathcal{T} \cap D\mathcal{O}_-(k)$ . Moreover,  $C_*$  is left adjoint to the inclusion functor*

$$i : D\mathcal{O}_-^{\text{eff}}(k) \rightarrow D\mathcal{O}_-(k)$$

*and  $D\mathcal{O}_-^{\text{eff}}(k)$  is triangle equivalent to the quotient category  $D\mathcal{O}_-(k) / \mathcal{T}_-$ .*

*Proof.* (1). The localization theory of compactly generated triangulated categories implies the quotient category  $SH_{S^1}^{\text{nis}} \mathcal{O} / \mathcal{T}$  is triangle equivalent to the full triangulated subcategory

$$\mathcal{T}^\perp = \{M \in SH_{S^1}^{\text{nis}} \mathcal{O} \mid \text{Hom}_{SH_{S^1}^{\text{nis}} \mathcal{O}}(T, M) = 0 \text{ for all } T \in \mathcal{T}\}.$$

Moreover,

$$\mathcal{T} = {}^\perp(\mathcal{T}^\perp) = \{X \in SH_{S^1}^{\text{nis}} \mathcal{O} \mid \text{Hom}_{SH_{S^1}^{\text{nis}} \mathcal{O}}(X, M) = 0 \text{ for all } M \in \mathcal{T}^\perp\}.$$

By construction,  $\mathcal{T}^\perp$  can be identified up to natural triangle equivalence with the full triangulated subcategory of  $\mathbb{A}^1$ -local  $\mathcal{O}$ -modules. The latter subcategory is naturally equivalent to  $SH_{S^1}^{\text{mot}} \mathcal{O}$ , because the motivic model structure on  $\mathcal{O}$ -modules is obtained from the Nisnevich local model structure by Bousfield localization with respect to the maps

$$\mathcal{O}(-, X \times \mathbb{A}^1) \rightarrow \mathcal{O}(-, X), \quad X \in Sm/k.$$

Recall that a map  $M \rightarrow N$  of  $\mathcal{O}$ -modules is a motivic equivalence if and only if for any  $\mathbb{A}^1$ -local  $\mathcal{O}$ -module  $L$  the induced map

$$\mathrm{Hom}_{SH_{S^1}^{\mathrm{nis}} \mathcal{O}}(N, L) \rightarrow \mathrm{Hom}_{SH_{S^1}^{\mathrm{nis}} \mathcal{O}}(M, L)$$

is an isomorphism. Given an  $\mathcal{O}$ -module  $M$ , the map  $M \rightarrow C_*(M)$  is a motivic equivalence by Theorem 3.2. If we fit the arrow into a triangle in  $SH_{S^1}^{\mathrm{nis}} \mathcal{O}$

$$(4) \quad X_M \rightarrow M \rightarrow C_*(M) \rightarrow X_M[1],$$

it will follow that  $\mathrm{Hom}_{SH_{S^1}^{\mathrm{nis}} \mathcal{O}}(X_M, L) = 0$  for all  $L \in \mathcal{T}^\perp$ . We see that for any  $\mathcal{O}$ -module  $M$  one has  $X_M \in {}^\perp(\mathcal{T}^\perp) = \mathcal{T}$ .

If  $C_*(M) \cong 0$  in  $SH_{S^1}^{\mathrm{nis}} \mathcal{O}$ , then  $M \cong X_M \in \mathcal{T}$ . Thus,  $M \in \mathcal{T}$  in this case. On the other hand, if  $M \in \mathcal{T}$  then  $C_*(M) \in \mathcal{T}$ , since  $X_M \in \mathcal{T}$  and  $\mathcal{T}$  is a thick triangulated subcategory in  $SH_{S^1}^{\mathrm{nis}} \mathcal{O}$ . On the other hand, Theorem 3.2 implies  $C_*(M) \in \mathcal{T}^\perp$ , and therefore  $C_*(M) \in \mathcal{T} \cap \mathcal{T}^\perp = 0$ . We conclude that  $\mathcal{T} = \mathrm{Ker} C_*$ .

(2). For any  $M \in \mathrm{Mod} \mathcal{O}$  the presheaves  $\pi_i(C_*(M))$ ,  $i \in \mathbb{Z}$ , are homotopy invariant and have  $\pi_0 \mathcal{O}$ -transfers. Since  $\mathcal{O}$  is a strict  $V$ -spectral category then each Nisnevich sheaf  $\pi_i^{\mathrm{nis}}(C_*(M))$  is homotopy invariant. Therefore the functor

$$C_* : D\mathcal{O}_-(k) \rightarrow D\mathcal{O}_-(k)$$

lands in  $D\mathcal{O}_-^{\mathrm{eff}}(k)$ . It follows from the first part of the theorem that the kernel of  $C_*$  is  $\mathcal{T}_- := \mathcal{T} \cap D\mathcal{O}_-(k)$ .

Let us prove that  $D\mathcal{O}_-^{\mathrm{eff}}(k) = \mathcal{T}^\perp \cap D\mathcal{O}_-(k)$ . Clearly,  $\mathcal{T}^\perp \cap D\mathcal{O}_-(k) \subset D\mathcal{O}_-^{\mathrm{eff}}(k)$ . Suppose  $M \in D\mathcal{O}_-^{\mathrm{eff}}(k)$ . Then  $M_f \in D\mathcal{O}_-^{\mathrm{eff}}(k)$ . We have that  $M_f$  is a fibrant  $\mathcal{O}$ -module in the Nisnevich local model structure and each  $\pi_i^{\mathrm{nis}}(M_f)$  is a strictly homotopy invariant sheaf, because  $\mathcal{O}$  is a strict  $V$ -spectral category. By [8, 6.2.7]  $M_f$  is  $\mathbb{A}^1$ -local in the motivic model category structure on  $\mathrm{Pre}^\Sigma(\mathrm{Sm}/k)$ . By Lemma 2.6  $\mathcal{O}$  is motivically excisive, hence [1, 5.12] implies  $M_f$  is  $\mathbb{A}^1$ -local in the motivic model category structure on  $\mathrm{Mod} \mathcal{O}$ . We see that  $M \in \mathcal{T}^\perp \cap D\mathcal{O}_-(k)$ .

Let  $E \in D\mathcal{O}_-^{\mathrm{eff}}(k)$  and  $M \in D\mathcal{O}_-(k)$ . Applying the functor  $\mathrm{Hom}_{D\mathcal{O}_-(k)}(-, E)$  to triangle (4), one gets

$$\mathrm{Hom}_{D\mathcal{O}_-(k)}(M, E) \cong \mathrm{Hom}_{D\mathcal{O}_-(k)}(C_*(M), E) = \mathrm{Hom}_{D\mathcal{O}_-^{\mathrm{eff}}(k)}(C_*(M), E).$$

Thus  $C_*$  is left adjoint to the inclusion functor  $i : D\mathcal{O}_-^{\mathrm{eff}}(k) \rightarrow D\mathcal{O}_-(k)$ .

It remains to show that  $D\mathcal{O}_-^{\mathrm{eff}}(k)$  is triangle equivalent to the quotient category  $D\mathcal{O}_-(k)/\mathcal{T}_-$ . By the first part of the theorem it is enough to prove that the natural functor

$$D\mathcal{O}_-(k)/\mathcal{T}_- \rightarrow SH_{S^1}^{\mathrm{nis}} \mathcal{O}/\mathcal{T}$$

is fully faithful. Consider an arrow  $M \xrightarrow{s} N$  in  $SH_{S^1}^{\mathrm{nis}} \mathcal{O}$ , where  $M \in D\mathcal{O}_-(k)$  and  $s$  is such that  $\mathrm{cone}(s) \in \mathcal{T}$ . There is a commutative diagram in  $SH_{S^1}^{\mathrm{nis}} \mathcal{O}$

$$\begin{array}{ccc} M & \xrightarrow{s} & N \\ u_M \downarrow & & \downarrow u_N \\ C_*(M) & \xrightarrow{C_*(s)} & C_*(N) \end{array}$$

in which cones of the vertical arrows are in  $\mathcal{T}$ . Since  $\mathrm{cone}(C_*(s)) \cong C_*(\mathrm{cone}(s)) = 0$  in  $SH_{S^1}^{\mathrm{nis}} \mathcal{O}$ , we see that  $C_*(s)$  is an isomorphism in  $SH_{S^1}^{\mathrm{nis}} \mathcal{O}$ . Therefore  $C_*(N) \in D\mathcal{O}_-(k)$  and  $\mathrm{cone}(u_N \circ s) \in \mathcal{T}_-$ . By [6, 9.1]  $D\mathcal{O}_-(k)/\mathcal{T}_-$  is a full subcategory of  $SH_{S^1}^{\mathrm{nis}} \mathcal{O}/\mathcal{T}$ .  $\square$



Let  $\mathcal{O}_{cor}$  be the Eilenberg–Mac Lane spectral category associated with the ringoid  $Cor$ . If the field  $k$  is perfect then [14] implies  $\mathcal{O}_{cor}$  is a strict  $V$ -spectral category. Recall that the Voevodsky triangulated category of motives  $DM_-^{eff}(k)$  is the full triangulated subcategory of (cohomologically) bounded above complexes of the derived category  $D(ShTr)$  of Nisnevich sheaves with transfers (see [12, 15]). The next result says that  $DM_-^{eff}(k)$  can be recovered from  $D\mathcal{O}_-^{eff}(k)$  if  $\mathcal{O} = \mathcal{O}_{cor}$ .

**Corollary 3.6.** *Let  $k$  be a perfect field and  $\mathcal{O} = \mathcal{O}_{cor}$ , then there is a natural triangle equivalence of triangulated categories*

$$D\mathcal{O}_-^{eff}(k) \xrightarrow{\sim} DM_-^{eff}(k).$$

*Proof.* By [1, section 6]  $SH_{S_1}^{nis}\mathcal{O}$  is naturally triangle equivalent to  $D(ShTr)$ . Moreover, this equivalence takes bounded below  $\mathcal{O}$ -modules to (cohomologically) bounded above complexes. Restriction of the equivalence to  $D\mathcal{O}_-^{eff}(k)$  yields the desired triangle equivalence of  $D\mathcal{O}_-^{eff}(k)$  and  $DM_-^{eff}(k)$ .  $\square$

To conclude the section, it is also worth to mention another way of constructing a motivic fibrant replacement on  $\mathcal{O}$ -modules. Namely, for any  $M \in \text{Mod } \mathcal{O}$  we set

$$\tilde{C}_*(M) := |d \mapsto (\underline{\text{Hom}}(\Delta^d, M))_f|.$$

Clearly,  $\tilde{C}_*(M)$  is functorial in  $M$ . Observe that if  $M$  is Nisnevich local then  $C_*(M)$  is zigzag level equivalent to  $\tilde{C}_*(M)$ , because  $\underline{\text{Hom}}(\Delta^d, M)$  and  $(\underline{\text{Hom}}(\Delta^d, M))_f$  are Nisnevich local and the arrows

$$\underline{\text{Hom}}(\Delta^d, M_f) \leftarrow \underline{\text{Hom}}(\Delta^d, M) \rightarrow (\underline{\text{Hom}}(\Delta^d, M))_f$$

are level weak equivalences.

**Proposition 3.7.** *The natural map  $M \rightarrow \tilde{C}_*(M)_f$  is an  $\mathbb{A}^1$ -local replacement of  $M$  in the motivic model structure of  $\mathcal{O}$ -modules.*

*Proof.* The map  $M \rightarrow \tilde{C}_*(M)_f$  is the composite

$$M \rightarrow |d \mapsto \underline{\text{Hom}}(\Delta^d, M)| \rightarrow |d \mapsto (\underline{\text{Hom}}(\Delta^d, M))_f| \rightarrow \tilde{C}_*(M)_f.$$

The left arrow is a level  $\mathbb{A}^1$ -weak equivalence in  $Pre^\Sigma(Sm/k)$  by [9, 3.8]. The middle arrow is a Nisnevich local weak equivalence, because it is the realization of a simplicial Nisnevich local weak equivalence. The right arrow is plainly a Nisnevich local weak equivalence as well.

The presheaves  $\pi_i(|d \mapsto \underline{\text{Hom}}(\Delta^d, M)|)$ ,  $i \in \mathbb{Z}$ , are homotopy invariant and have  $\pi_0\mathcal{O}$ -transfers. Since  $\mathcal{O}$  is a strict  $V$ -spectral category then each Nisnevich sheaf  $\pi_i^{nis}(\tilde{C}_*(M)_f)$  is strictly homotopy invariant and has  $\pi_0\mathcal{O}$ -transfers. By [8, 6.2.7]  $C_*(M)_f$  is  $\mathbb{A}^1$ -local in the motivic model category structure on  $Pre^\Sigma(Sm/k)$ . By Lemma 2.6  $\mathcal{O}$  is motivically excisive, hence [1, 5.12] implies the arrow of the proposition is an  $\mathbb{A}^1$ -weak equivalence in  $\text{Mod } \mathcal{O}$ .  $\square$

#### 4. THE SPECTRAL CATEGORY $\mathbb{K}$

In this section the definition of the  $V$ -spectral category  $\mathbb{K}$  is given. It is obtained by taking  $K$ -theory symmetric spectra  $K(\mathcal{A}(U, X))$  of certain additive categories  $\mathcal{A}(U, X)$ ,  $U, X \in Sm/k$ . To define these categories we need some preliminaries.

**Notation 4.1.** Let  $U, X \in Sm/k$ . Define  $\text{Supp}(U \times X/X)$  as the set of all closed subsets in  $U \times X$  of the form  $A = \cup A_i$ , where each  $A_i$  is a closed irreducible subset in  $U \times X$  which is finite and surjective over  $U$ . The empty subset in  $U \times X$  is also regarded as an element of  $\text{Supp}(U \times X/X)$ .

**Notation 4.2.** Given  $U, X \in \mathcal{S}m/k$  and  $A \in \text{Supp}(U \times X/X)$ , let  $I_A \subset \mathcal{O}_{U \times X}$  be the ideal sheaf of the closed set  $A \subset U \times X$ . Denote by  $A_m$  the closed subscheme in  $U \times X$  of the form  $(A, \mathcal{O}_{U \times X}/I_A^m)$ . If  $m = 0$ , then  $A_m$  is the empty subscheme. Define  $\text{SubSch}(U \times X/X)$  as the set of all closed subschemes in  $U \times X$  of the form  $A_m$ .

For any  $Z \in \text{SubSch}(U \times X/X)$  we write  $p_U^Z : Z \rightarrow U$  to denote  $p \circ i$ , where  $i : Z \hookrightarrow U \times X$  is the closed embedding and  $p : U \times X \rightarrow U$  is the projection. If there is no likelihood of confusion we shall write  $p_U$  instead of  $p_U^Z$ , dropping  $Z$  from notation.

Clearly, for any  $Z \in \text{SubSch}(U \times X/X)$  the reduced scheme  $Z^{\text{red}}$ , regarded as a closed subset of  $U \times X$ , belongs to  $\text{Supp}(U \times X/X)$ .

**Notation 4.3.** Let  $V, U, X \in \mathcal{S}m/k$ . Let  $A \in \text{Supp}(V \times U/U)$ ,  $B \in \text{Supp}(U \times X/X)$ . Set

$$B \circ A = p_{VX}(V \times B \cap A \times X) \subset V \times X,$$

where  $p_{VX} : V \times U \times X \rightarrow V \times X$  is the projection. One can check that

$$B \circ A \in \text{Supp}(V \times X/X).$$

**Notation 4.4.** Let  $V, U, X \in \mathcal{S}m/k$ . Let  $S \in \text{SubSch}(V \times U/U)$ ,  $Z \in \text{SubSch}(U \times X/X)$ . By 4.2 one has  $S^{\text{red}} \in \text{Supp}(V \times U/U)$ ,  $Z^{\text{red}} \in \text{Supp}(U \times X/X)$ . By 4.3 one has  $Z^{\text{red}} \circ S^{\text{red}} \in \text{Supp}(V \times X/X)$ . One can show that for some integer  $k \gg 0$  there exists a scheme morphism  $\pi_k : T = S \times X \cap V \times Z \rightarrow (Z^{\text{red}} \circ S^{\text{red}})_k$  such that  $i_k \circ \pi_k = p_{VX} \circ i_T : T \rightarrow V \times X$ , where  $i_k : (Z^{\text{red}} \circ S^{\text{red}})_k \hookrightarrow V \times X$  and  $i_T : T \hookrightarrow V \times U \times X$  are closed embeddings,  $p_{VX} : V \times U \times X \rightarrow V \times X$  is the projection.

If there exists  $\pi_k$  satisfying the condition above then it is unique. Moreover, for any  $m > k$  one has  $i_k^m \circ \pi_k = \pi_m$ , where  $i_k^m : (Z^{\text{red}} \circ S^{\text{red}})_k \hookrightarrow (Z^{\text{red}} \circ S^{\text{red}})_m$  is the closed embedding.

We shall often write  $Z \circ S$  to denote  $(Z^{\text{red}} \circ S^{\text{red}})_k$ , provided that there exists the required  $\pi_k$ . In this case we shall also write  $\pi$  to denote  $\pi_k : T \rightarrow (Z \circ S)$ .

**Definition 4.5.** For any  $U, X \in \mathcal{S}m/k$  we define objects of  $\mathcal{A}(U, X)$  as equivalence classes for the triples

$$(n, Z, \varphi : p_{U,*}(\mathcal{O}_Z) \rightarrow M_n(\mathcal{O}_U)),$$

where  $n$  is a nonnegative integer,  $Z \in \text{SubSch}(U \times X/X)$  and  $\varphi$  is a non-unital homomorphism of sheaves of  $\mathcal{O}_U$ -algebras. Let  $p(\varphi)$  be the idempotent  $\varphi(1) \in M_n(\Gamma(U, \mathcal{O}_U))$ , then  $P(\varphi) = \text{Im}(p(\varphi))$  can be regarded as a  $p_{U,*}(\mathcal{O}_Z)$ -module by means of  $\varphi$ .

By definition, two triples  $(n, Z, \varphi)$ ,  $(n', Z', \varphi')$  are equivalent if  $n = n'$  and there is a triple  $(n'', Z'', \varphi'')$  such that  $n = n' = n''$ ,  $Z, Z' \subset Z''$  are closed subschemes in  $Z''$ , and the diagrams

$$\begin{array}{ccc} p_{U,*}(\mathcal{O}_Z) & \xrightarrow{\varphi} & M_n(\mathcal{O}_U) \\ & \nwarrow \text{can} \quad \nearrow \varphi'' & \\ & p_{U,*}(\mathcal{O}_{Z''}) & \end{array} \quad \begin{array}{ccc} p_{U,*}(\mathcal{O}_{Z'}) & \xrightarrow{\varphi'} & M_n(\mathcal{O}_U) \\ & \nwarrow \text{can} \quad \nearrow \varphi'' & \\ & p_{U,*}(\mathcal{O}_{Z''}) & \end{array}$$

are commutative. We shall often denote an equivalence class for the triples by  $\Phi$ . Though  $Z$  is not uniquely defined by  $\Phi$ , nevertheless we shall also refer to  $Z \subset U \times X$  as the *support* of  $\Phi$ .

Given  $\Phi, \Phi' \in \mathcal{A}(U, X)$  we first equalize supports  $Z, Z'$  of the objects  $\Phi, \Phi'$  and then set

$$\text{Hom}_{\mathcal{A}(U, X)}(\Phi, \Phi') = \text{Hom}_{p_{U,*}(\mathcal{O}_Z)}(P(\varphi), P(\varphi')).$$

Given any three objects  $\Phi, \Phi', \Phi'' \in \mathcal{A}(U, X)$  a composition law

$$\text{Hom}_{\mathcal{A}(U, X)}(\Phi, \Phi') \circ \text{Hom}_{\mathcal{A}(U, X)}(\Phi', \Phi'') \rightarrow \text{Hom}_{\mathcal{A}(U, X)}(\Phi, \Phi'')$$

is defined in the obvious way. This makes therefore  $\mathcal{A}(U, X)$  an additive category. The zero object is the equivalence class of the triple  $(0, \emptyset, 0)$ . By definition,

$$\Phi_1 \oplus \Phi_2 = (n_1 + n_2, Z_1 \cup Z_2, p_{U,*}(\mathcal{O}_{Z_1 \cup Z_2}) \rightarrow p_{U,*}(\mathcal{O}_{Z_1}) \times p_{U,*}(\mathcal{O}_{Z_2}) \rightarrow M_{n_1}(\mathcal{O}_U) \times M_{n_2}(\mathcal{O}_U)) \hookrightarrow M_{n_1+n_2}(\mathcal{O}_U).$$

Clearly,  $P(\varphi_1 \oplus \varphi_2) \cong P(\varphi_1) \oplus P(\varphi_2)$ .

We now want to construct a bilinear pairing

$$(5) \quad \mathcal{A}(V, U) \times \mathcal{A}(U, X) \xrightarrow{\circ} \mathcal{A}(V, X), \quad U, V, X \in \text{Sm}/k.$$

First, define it on objects. Namely,

$$((n_1, Z_1, \varphi_1), (n_2, Z_2, \varphi_2)) \mapsto (n_1 n_2, Z_2 \circ Z_1, \varphi_2 \circ \varphi_1),$$

where  $Z_2 \circ Z_1 \in \text{SubSch}(V \times X/X)$  is a closed subscheme of  $V \times X$  defined in Notation 4.4. The nonunital homomorphism  $\varphi_2 \circ \varphi_1 : p_{V,*}^{(Z_2 \circ Z_1)}(\mathcal{O}_{Z_2 \circ Z_1}) \rightarrow M_{n_2 n_1}(\mathcal{O}_V)$  is given by the composition

$$(6) \quad \begin{array}{ccc} & & M_{n_2}(M_{n_1}(\mathcal{O}_V)) \xrightarrow{L} M_{n_2 n_1}(\mathcal{O}_V) \\ & & \uparrow M_{n_2}(\varphi_1) \\ q_{V,*}(\mathcal{O}_{Z_1 \times_U Z_2}) = p_{V,*}(p_{Z_1,*}(\mathcal{O}_{Z_1 \times_U Z_2})) & \xrightarrow{p_{V,*}(\varphi_{2,Z_1})} & M_{n_2}(p_{V,*}(\mathcal{O}_{Z_1})) \\ \parallel & & \\ p_{V,*}^{(Z_2 \circ Z_1)}(\pi_*(\mathcal{O}_{Z_1 \times_U Z_2})) & & \\ \uparrow p_{V,*}^{Z_2 \circ Z_1}(\pi^*) = \text{can}' & & \\ p_{V,*}^{(Z_2 \circ Z_1)}(\mathcal{O}_{Z_2 \circ Z_1}) & & \end{array}$$

where  $L$  is a canonical isomorphism obtained by inserting  $(n_1, n_1)$ -matrices into entries of a  $(n_2, n_2)$ -matrix, the diagrams

$$\begin{array}{ccc} Z_1 \times_U Z_2 & \longrightarrow & Z_2 \longrightarrow X \\ \downarrow p_{Z_1} & & \downarrow p_U \\ q_V \downarrow Z_1 & \xrightarrow{r} & U \\ \downarrow p_V & & \\ V & & \end{array} \quad \begin{array}{ccc} Z_1 \times_U Z_2 & & \\ \downarrow q_V & \searrow \pi & \\ & V \times X & \longleftarrow Z_2 \circ Z_1 \\ & \swarrow p_V^{Z_2 \circ Z_1} & \end{array}$$

are commutative, and  $\pi^* : \mathcal{O}_{Z_2 \circ Z_1} \rightarrow \pi_*(\mathcal{O}_{Z_1 \times_U Z_2})$  is induced by the scheme morphism  $\pi : Z_1 \times_U Z_2 \rightarrow Z_2 \circ Z_1$  from Notation 4.4. Finally,  $\varphi_{2,Z_1} : p_{Z_1,*}(\mathcal{O}_{Z_1 \times_U Z_2}) \rightarrow M_{n_2}(\mathcal{O}_{Z_1})$  is defined as a unique non-unital homomorphism of sheaves of  $\mathcal{O}_{Z_1}$ -algebras such that for any open affine  $U' \subset U$  and any open affine  $Z'_1 \subset Z_1$  with  $r(Z'_1) \subset U'$  and  $Z'_2 = p_U^{-1}(U')$  the value of  $\varphi_{2,Z_1}$  on  $Z'_1$  coincides with the non-unital homomorphism of  $k[Z'_1]$ -algebras

$$k[Z'_1] \otimes_{k[U']} k[Z'_2] \xrightarrow{\text{id} \otimes \varphi_2} k[Z'_1] \otimes_{k[U']} M_{n_2}(k[U']) \xrightarrow{a \otimes \beta \mapsto a \cdot r^*(\beta)} M_{n_2}(k[Z'_1]).$$

For a future use set  $p(\varphi_{2,Z_1}) = \varphi_{2,Z_1}(1) \in M_{n_2}(\Gamma(Z_1, \mathcal{O}_{Z_1}))$  and  $P(\varphi_{2,Z_1}) = \text{Im}[p(\varphi_{2,Z_1}) : \mathcal{O}_{Z_1}^{n_2} \rightarrow \mathcal{O}_{Z_1}^{n_2}]$ .

In order to define pairing (5) on morphisms, we need some preparations. Let  $\Phi_1 \in \mathcal{A}(V, U)$  and  $\Phi_2 \in \mathcal{A}(U, X)$ . Consider the diagram

$$\begin{array}{ccccc}
p_{V,*}(\mathcal{O}_{Z_1}^{n_2}) \otimes_{p_{V,*}(\mathcal{O}_{Z_1})} P(\varphi_1) & \xrightarrow[\cong]{can} & P(\varphi_1)^{n_2} & \xrightarrow{i_1} & (\mathcal{O}_V^{n_1})^{n_2} & \xrightarrow[\cong]{\ell} & \mathcal{O}_V^{n_2 n_1} \\
\uparrow p_{V,*}(i_{2,Z_1}) \otimes \text{id} & & & & & & \uparrow i(\varphi_2 \circ \varphi_1) \downarrow p(\varphi_2 \circ \varphi_1) \\
p_{V,*}(P(\varphi_{2,Z_1})) \otimes_{p_{V,*}(\mathcal{O}_{Z_1})} P(\varphi_1) & \xrightarrow{\sigma_{12}} & P(\varphi_2 \circ \varphi_1), & & & & 
\end{array}$$

where  $\sigma_{12} = p(\varphi_2 \circ \varphi_1) \circ \ell \circ i_1 \circ can \circ (i_{2,Z_1} \otimes \text{id})$  (here  $\ell(e_{i,j}) = e_{i+(j-1)n_1}$ ). It is worth to note that the isomorphism  $\ell$  induces an  $\mathcal{O}_V$ -algebra isomorphism  $M_{n_2}(M_{n_1}(\mathcal{O}_V)) \cong M_{n_2 n_1}(\mathcal{O}_V)$  which coincides with the canonical isomorphism  $L$  obtained by inserting  $(n_1, n_1)$ -matrices into entries of a  $(n_2, n_2)$ -matrix.

**Definition 4.6.** An  $p_{V,*}^{(Z_2 \circ Z_1)}(\mathcal{O}_{Z_2 \circ Z_1})$ -module structure on  $p_{V,*}(P(\varphi_{2,Z_1})) \otimes_{p_{V,*}(\mathcal{O}_{Z_1})} P(\varphi_1)$  is defined as follows. For any open  $V^0 \subset V$ ,  $f \in \Gamma(V^0, p_{V,*}^{(Z_2 \circ Z_1)}(\mathcal{O}_{Z_2 \circ Z_1}))$ ,  $m_1 \in \Gamma(V^0, P(\varphi_1))$ , and  $m_2 \in \Gamma(V^0, p_{V,*}(P(\varphi_{2,Z_1})))$  set

$$f(m_2 \otimes m_1) = ((p_{V,*}(\varphi_{2,Z_1}) \circ can')(f))(m_2) \otimes m_1.$$

An  $p_{V,*}^{(Z_2 \circ Z_1)}(\mathcal{O}_{Z_2 \circ Z_1})$ -module structure on  $P(\varphi_2 \circ \varphi_1)$  is defined as follows. For any open  $V^0 \subset V$ ,  $f \in \Gamma(V^0, p_{V,*}^{(Z_2 \circ Z_1)}(\mathcal{O}_{Z_2 \circ Z_1}))$ , and  $m \in \Gamma(V^0, P(\varphi_2 \circ \varphi_1))$  set

$$fm = ((\varphi_2 \circ \varphi_1)(f))(m).$$

In particular,

$$1 \cdot m = ((\varphi_2 \circ \varphi_1)(1))(m) = p(\varphi_2 \circ \varphi_1)(m) = m,$$

because  $m \in \text{Im}(p(\varphi_2 \circ \varphi_1))$ .

**Lemma 4.7.** The map  $\sigma_{12}$  is an isomorphism of  $\mathcal{O}_V$ -modules and, moreover, an isomorphism of the  $p_{V,*}^{(Z_2 \circ Z_1)}(\mathcal{O}_{Z_2 \circ Z_1})$ -modules.

Let  $\alpha_1 : \Phi_1 \rightarrow \Phi'_1$  and  $\alpha_2 : \Phi_2 \rightarrow \Phi'_2$  be morphism in  $\mathcal{A}(V, U)$  and  $\mathcal{A}(U, X)$  respectively. We set

$$(7) \quad \alpha_2 \odot \alpha_1 = \sigma'_{12} \circ (\alpha_2 \otimes \alpha_1) \circ \sigma_{12}^{-1} : P(\varphi_2 \circ \varphi_1) \rightarrow P(\varphi'_2 \circ \varphi'_1).$$

The definition of pairing (5) is finished. It is defined on objects above and on morphisms by formula (7).

**Lemma 4.8.** The functor  $\mathcal{A}(V, U) \times \mathcal{A}(U, X) \xrightarrow{\circ} \mathcal{A}(V, X)$  is bilinear for all  $U, V, X \in \text{Sm}/k$ .

For any  $X \in \text{Sm}/k$  we define an object  $\text{id}_X \in \text{Ob } \mathcal{A}(X, X)$  by

$$\text{id}_X = (1, \Delta_X, \text{id} : \mathcal{O}_X \rightarrow \mathcal{O}_X).$$

**Lemma 4.9.** For any  $U, X \in \text{Sm}/k$  the functors  $\{\text{id}_U\} \times \mathcal{A}(U, X) \xrightarrow{\circ} \mathcal{A}(U, X)$  and  $\mathcal{A}(U, X) \times \{\text{id}_X\} \xrightarrow{\circ} \mathcal{A}(U, X)$  are identities on  $\mathcal{A}(U, X)$ .

**Lemma 4.10.** For any  $U, V, W, X \in \text{Sm}/k$  and any  $\Phi_1 \in \mathcal{A}(W, V), \Phi_2 \in \mathcal{A}(V, U), \Phi_3 \in \mathcal{A}(U, X)$  the following statements are true:

- (1)  $\Phi_3 \circ (\Phi_2 \circ \Phi_1) = (\Phi_3 \circ \Phi_2) \circ \Phi_1 \in \text{Ob } \mathcal{A}(W, X)$ ;
- (2)  $p(\Phi_3 \circ (\Phi_2 \circ \Phi_1)) = p((\Phi_3 \circ \Phi_2) \circ \Phi_1)$  and  $P(\Phi_3 \circ (\Phi_2 \circ \Phi_1)) = P((\Phi_3 \circ \Phi_2) \circ \Phi_1)$ ;
- (3) suppose  $\alpha_i : \Phi_i \rightarrow \Phi'_i$  are morphisms ( $i = 1, 2, 3$ ), then  $\alpha_3 \odot (\alpha_2 \odot \alpha_1) = (\alpha_3 \odot \alpha_2) \odot \alpha_1 \in \text{Hom}_{\mathcal{A}(W, X)}(P(\Phi_3 \circ (\Phi_2 \circ \Phi_1)), P((\Phi'_3 \circ \Phi'_2) \circ \Phi'_1))$ .

**Proposition 4.11.** For any  $U, V, W, X \in Sm/k$  the diagram of functors

$$\begin{array}{ccc}
 & (\mathcal{A}(W, V) \times \mathcal{A}(V, U)) \times \mathcal{A}(U, X) & \xrightarrow{\circ \times \text{id}} \mathcal{A}(W, U) \times \mathcal{A}(U, X) \\
 \nearrow \cong & & \downarrow \circ \\
 \mathcal{A}(W, V) \times (\mathcal{A}(V, U) \times \mathcal{A}(U, X)) & & \mathcal{A}(W, X) \\
 \searrow \text{id} \times \circ & \mathcal{A}(W, V) \times \mathcal{A}(V, X) & \xrightarrow{\circ} \mathcal{A}(W, X)
 \end{array}$$

is strictly commutative.

**Lemma 4.12.** Pairings (5) together with  $\{\text{id}_X\}_{X \in Sm/k}$  determine a category  $\mathcal{A}$  on  $Sm/k$  which is also enriched over additive categories. Moreover, the rules  $X \mapsto X$  and  $f \mapsto \Phi_f = (1, \Gamma_f, \text{id} : \mathcal{O}_U \rightarrow \mathcal{O}_U)$  give a functor  $\sigma : Sm/k \rightarrow \mathcal{A}$ .

The following notation will be useful later.

**Notation 4.13.** Let  $X, X', Y \in Sm/k$  and  $f : X' \rightarrow X$  be a morphism in  $Sm/k$ . Define a functor  $f^* : \mathcal{A}(X, Y) \rightarrow \mathcal{A}(X', Y)$  as the additive functor

$$\mathcal{A}(X, Y) \xrightarrow{- \circ \sigma(f)} \mathcal{A}(X', Y).$$

More precisely,  $f^*(\Phi) = \Phi \circ \sigma(f)$  and  $f^*(\alpha) = \alpha \odot \text{id}_{\sigma(f)}$ .

Let  $X, Y, Y' \in Sm/k$  and  $g : Y \rightarrow Y'$  be a morphism in  $Sm/k$ . Define a functor  $g_* : \mathcal{A}(X, Y) \rightarrow \mathcal{A}(X, Y')$  as the additive functor

$$\mathcal{A}(X, Y) \xrightarrow{\sigma(g) \circ -} \mathcal{A}(X, Y').$$

Namely,  $g_*(\Phi) = \sigma(g) \circ \Phi$  and  $g_*(\alpha) = \text{id}_{\sigma(g)} \odot \alpha$ .

Using this notation and Proposition 4.11, one has the following

**Corollary 4.14.** Let  $f : X' \rightarrow X$  and  $g : Y \rightarrow Y'$  be morphisms in  $Sm/k$ . Then  $f^* \circ g_* = g_* \circ f^* : \mathcal{A}(X, Y) \rightarrow \mathcal{A}(X', Y')$ . If  $f' : X'' \rightarrow X'$  is a map in  $Sm/k$  then  $(f \circ f')^* = (f')^* \circ f^* : \mathcal{A}(X, Y) \rightarrow \mathcal{A}(X'', Y)$ . Also, for any map  $g' : Y' \rightarrow Y''$  in  $Sm/k$  one has  $(g' \circ g)_* = (g')_* \circ g_* : \mathcal{A}(X, Y) \rightarrow \mathcal{A}(X, Y'')$ .

By [3, §6.1] for an additive category  $\mathcal{C}$ , one can define the structure of a symmetric spectrum on the Waldhausen  $K$ -theory spectrum  $K(\mathcal{C})$ . By definition,

$$K(\mathcal{C})_n = |\text{Obs} S^{\mathcal{Q}} \mathcal{C}|, \quad \mathcal{Q} = \{1, \dots, n\}.$$

Moreover, strictly associative bilinear pairings of additive categories induce strictly associative pairings of their  $K$ -theory symmetric spectra (see [3, §6.1]).

Consider the category  $\mathcal{A}$  on  $Sm/k$ . Given  $U, X \in Sm/k$ , one sets

$$\mathbb{K}(U, X) := K(\mathcal{A}(U, X)).$$

Pairing (5) yields a pairing of symmetric spectra

$$(8) \quad \mathbb{K}(V, U) \wedge \mathbb{K}(U, X) \rightarrow \mathbb{K}(V, X).$$

Proposition 4.11 implies that (8) is a strictly associative pairing. Moreover, for any  $X \in Sm/k$  there is a map  $\mathbf{1} : S \rightarrow \mathbb{K}(X, X)$  which is subject to the unit coherence law (see [3, section 6.1]). Note that  $\mathbf{1}_0 : S^0 \rightarrow \mathbb{K}(X, X)_0$  is the map which sends the basepoint to the null object and the non-basepoint to the unit object  $\text{id}_X$ .

Thus we get the following

**Theorem 4.15.** *The triple  $(\mathbb{K}, \wedge, \mathbf{1})$  determines a spectral category on  $Sm/k$ . Moreover, the functor  $\sigma : Sm/k \rightarrow \mathcal{A}$  of Lemma 4.12 gives a spectral functor*

$$\sigma : \mathcal{O}_{naive} \rightarrow \mathbb{K}$$

between spectral categories.

We now want to define a spectral functor

$$\square : \mathbb{K} \wedge \mathcal{O}_{naive} \rightarrow \mathbb{K}.$$

It is in fact determined by additive functors

$$f^* : \mathcal{A}(X, X') \rightarrow \mathcal{A}(X \times U, X' \times U'), \quad f : U \rightarrow U' \in \text{Mor}(Sm/k),$$

satisfying certain reasonable properties mentioned below. If

$$(n, Z, \varphi : p_{X,*}^Z(\mathcal{O}_Z) \rightarrow M_n(\mathcal{O}_X))$$

is a representative for  $\Phi \in \mathcal{A}(X, X')$ , then  $f^*(\Phi)$  is represented by the triple

$$(n, Z \times \Gamma_f, \varphi \boxtimes \text{id}_U : (p_X^Z \times \text{id})_*(\mathcal{O}_{Z \times \Gamma_f}) \rightarrow M_n(\mathcal{O}_{X \times U})).$$

Here  $\varphi \boxtimes \text{id}_U$  is a unique non-unital homomorphism of sheaves of  $\mathcal{O}_{X \times U}$ -algebras such that for any affine open subsets  $X_0 \subset X$ ,  $U_0 \subset U$  and for  $Z_0 = (p_X^Z)^{-1}(X_0) \subset Z$  the value of  $\varphi \boxtimes \text{id}_U$  on  $X_0 \times U_0$  is the following non-unital homomorphism of  $k[Z_0 \times U_0]$ -algebras:

$$(\varphi \boxtimes \text{id}_U)(a \boxtimes b) := (q_{X,0})^*(\varphi(a)) \cdot (q_{U,0})^*(b) \in M_n(k[X_0 \times U_0]),$$

where  $q_{X,0} : X^0 \times U^0 \rightarrow X^0$  and  $q_{U,0} : X^0 \times U^0 \rightarrow U^0$  are the projections.

To define  $f^*$  on morphisms, we note that the canonical morphism

$$adj : q_X^*(P(\varphi)) \xrightarrow{q_X^*(ip(\varphi))} q_X^*(\mathcal{O}_X^n) \xrightarrow{can} \mathcal{O}_{X \times U}^n \xrightarrow{p(f^*(\Phi))} P(f^*(\Phi))$$

is an isomorphism. Given a morphism  $\alpha : \Phi \rightarrow \Phi'$  in  $\mathcal{A}(X, X')$ , we set

$$f^*(\alpha) = adj' \circ q_X^*(\alpha) \circ adj^{-1} : P(f^*(\Phi)) \rightarrow P(f^*(\Phi')).$$

Clearly,  $f^*$  is an additive functor.

**Proposition 4.16.** *Let  $f_1 : U \rightarrow U'$ ,  $f_2 : U' \rightarrow U''$  be two maps in  $Sm/k$ ,  $\Phi_1, \Phi'_1 \in \text{Ob } \mathcal{A}(X, X')$ ,  $\Phi_2, \Phi'_2 \in \text{Ob } \mathcal{A}(X', X'')$ , let  $\alpha_1 : \Phi_1 \rightarrow \Phi'_1$  be a morphism in  $\mathcal{A}(X, X')$  and let  $\alpha_2 : \Phi_2 \rightarrow \Phi'_2$  be a morphism in  $\mathcal{A}(X', X'')$ . Then*

- (1)  $(f_2 \circ f_1)^*(\Phi_2 \circ \Phi_1) = f_2^*(\Phi_2) \circ f_1^*(\Phi_1);$
- (2)  $(f_2 \circ f_1)^*(\alpha_2 \circ \alpha_1) = f_2^*(\alpha_2) \circ f_1^*(\alpha_1).$

**Corollary 4.17.** *Under the assumptions of Proposition 4.16 the diagram of functors*

$$\begin{array}{ccc} \mathcal{A}(X, X') \times \mathcal{A}(X', X'') & \xrightarrow{\circ} & \mathcal{A}(X, X'') \\ f_1^* \times f_2^* \downarrow & & \downarrow (f_2 \circ f_1)^* \\ \mathcal{A}(X \times U, X' \times U') \times \mathcal{A}(X' \times U', X'' \times U'') & \xrightarrow{\circ} & \mathcal{A}(X \times U, X'' \times U'') \end{array}$$

is commutative.

**Corollary 4.18.** *We have a spectral functor*

$$\square : \mathbb{K} \wedge \mathcal{O}_{naive} \rightarrow \mathbb{K}$$

such that  $(X, U) \in Sm/k \times Sm/k$  is mapped to  $X \times U \in Sm/k$ . Moreover, for any morphism  $h : X \rightarrow X'$  in  $Sm/k$ , regarded as the object  $\sigma(h)$  of  $\mathcal{A}(X, X')$ , one has

$$f^*(\sigma(h)) = \sigma(f \times h) \in \text{Ob } \mathcal{A}(X \times U, X' \times U')$$

for every morphism of  $k$ -smooth schemes  $f : U \rightarrow U'$ .

In what follows we shall denote by  $\mathbb{K}_0$  the ringoid  $\pi_0(\mathbb{K})$ .

**Theorem 4.19** (Knizel [7]). *For any  $\mathbb{K}_0$ -presheaf of abelian groups  $\mathcal{F}$ , i.e.  $\mathcal{F}$  is a contravariant functor from  $\mathbb{K}_0$  to abelian groups, the associated Nisnevich sheaf  $\mathcal{F}_{\text{nis}}$  has a unique structure of a  $\mathbb{K}_0$ -presheaf for which the canonical homomorphism  $\mathcal{F} \rightarrow \mathcal{F}_{\text{nis}}$  is a homomorphism of  $\mathbb{K}_0$ -presheaves. If  $\mathcal{F}$  is homotopy invariant then so is  $\mathcal{F}_{\text{nis}}$ . Moreover, if the field  $k$  is perfect then every  $\mathbb{A}^1$ -invariant Nisnevich  $\mathbb{K}_0$ -sheaf is strictly  $\mathbb{A}^1$ -invariant.*

**Remark 4.20.** Although the category  $\mathcal{A}(X, Y)$  is different from the category of bimodules  $\mathcal{P}(X, Y)$  (see Appendix for the definition of  $\mathcal{P}(X, Y)$ ), the proof of the preceding theorem is in spirit similar to the proof of the same fact for  $K_0$ -presheaves obtained by Walker [17].

**Proposition 4.21.**  $\mathbb{K}_0$  is a  $V$ -ringoid. If the field  $k$  is perfect then it is also a strict  $V$ -ringoid.

*Proof.* The proof of [1, 5.9] shows that for any elementary distinguished square the sequence of Nisnevich sheaves associated to representable presheaves

$$0 \rightarrow \mathbb{K}_{0, \text{nis}}(-, U') \rightarrow \mathbb{K}_{0, \text{nis}}(-, U) \oplus \mathbb{K}_{0, \text{nis}}(-, X') \rightarrow \mathbb{K}_{0, \text{nis}}(-, X) \rightarrow 0$$

is exact.

Let  $\rho : Sm/k \rightarrow \mathbb{K}_0$  be the composite functor

$$(9) \quad Sm/k \rightarrow \pi_0 \mathcal{O}_{naive} \xrightarrow{\pi_0(\sigma)} \pi_0(\mathbb{K}) = \mathbb{K}_0,$$

where  $\sigma : \mathcal{O}_{naive} \rightarrow \mathbb{K}$  is the spectral functor constructed in Theorem 4.15. Also, let a functor  $\boxtimes : \mathbb{K}_0 \times Sm/k \rightarrow \mathbb{K}_0$  be the composite functor

$$(10) \quad \mathbb{K}_0 \times Sm/k \rightarrow \mathbb{K}_0 \times \pi_0 \mathcal{O}_{naive} \rightarrow \pi_0(\mathbb{K} \wedge \mathcal{O}_{naive}) \xrightarrow{\pi_0(\square)} \mathbb{K}_0,$$

where  $\square : \mathbb{K} \wedge \mathcal{O}_{naive} \rightarrow \mathbb{K}$  is the spectral functor constructed in Corollary 4.18. Then we have that  $\text{id}_X \boxtimes f = \rho(\text{id}_X \times f)$ ,  $(u + v) \boxtimes f = u \boxtimes f + v \boxtimes f$  for all  $u, v \in \text{Mor}(\mathbb{K}_0)$  and  $f \in \text{Mor}(Sm/k)$ .

Theorem 4.19 now implies  $\mathbb{K}_0$  is a  $V$ -ringoid. It also follows from Theorem 4.19 that it is a strict  $V$ -ringoid over perfect fields.  $\square$

We are now in position to prove the main result of the section.

**Theorem 4.22.** *The spectral category  $\mathbb{K}$  is a  $V$ -spectral category. If the field  $k$  is perfect then it is also a strict  $V$ -spectral category.*

*Proof.*  $\mathbb{K}$  is connective by construction. It is proved similar to [1, 5.9] that  $\mathbb{K}$  is Nisnevich excisive. The structure spectral functor

$$\sigma : \mathcal{O}_{naive} \rightarrow \mathbb{K}$$

is constructed in Theorem 4.15.

It follows from Corollary 4.18 that there is a spectral functor

$$\square : \mathbb{K} \wedge \mathcal{O}_{naive} \rightarrow \mathbb{K}$$

sending  $(X, U) \in Sm/k \times Sm/k$  to  $X \times U \in Sm/k$  and such that  $\text{id}_X \square f = \sigma(\text{id}_X \times f)$  for all  $f \in \text{Mor}(Sm/k)$ . Proposition 4.21 implies the ringoid  $\mathbb{K}_0$  together with structure functors (9) and (10) is a  $V$ -ringoid which is strict whenever the base field  $k$  is perfect.  $\square$

We are now able to introduce the triangulated category of  $K$ -motives.

**Definition 4.23.** Suppose  $k$  is a perfect field. The *triangulated category of  $K$ -motives*  $DK_-^{eff}(k)$  is the triangulated category  $D\mathcal{O}_-^{eff}(k)$  constructed in Section 3 associated to the strict  $V$ -spectral category  $\mathcal{O} = \mathbb{K}$  of Theorem 4.22.

To conclude the section, we discuss further useful properties of categories  $\mathcal{A}(U, X)$ -s.

**Proposition 4.24.** Under Notation 4.13 and the notation of Lemma 4.12 and the notation which are just above Proposition 4.16 for any  $X, Y \in Sm/k$  and any morphism  $f : U \rightarrow V$  in  $Sm/k$  the following square of additive functors is strictly commutative

$$\begin{array}{ccc} \mathcal{A}(X \times V, Y \times V) & \xrightarrow{(1_X \times f)^*} & \mathcal{A}(X \times U, Y \times V) \\ \uparrow \text{id}_V^* & & \uparrow (1_Y \times f)^* \\ \mathcal{A}(X, Y) & \xrightarrow{\text{id}_U^*} & \mathcal{A}(X \times U, Y \times U). \end{array}$$

**Notation 4.25.** For every  $X \in Sm/k, Y \in Sm/k$  and  $n > 0$ , denote by  $\mathcal{A}(X, Y)(\mathbb{G}_m^{\times n})$  the category whose objects are the tuples  $(\Phi, \theta_1, \dots, \theta_n)$ , where  $\Phi \in \mathcal{A}(X, Y)$  and  $(\theta_1, \dots, \theta_n)$  are commuting automorphisms of  $\Phi$ . Morphisms from  $(\Phi, \theta_1, \dots, \theta_n)$  to  $(\Phi', \theta'_1, \dots, \theta'_n)$  are given by morphisms  $\alpha : \Phi \rightarrow \Phi'$  in  $\mathcal{A}(X, Y)$  such that  $\alpha \circ \theta_i = \theta'_i \circ \alpha$  for every  $i = 1, \dots, n$ .

Using Notation 4.13 for a morphism  $f : X' \rightarrow X$  in  $Sm/k$ , define an additive functor

$$f_n^* : \mathcal{A}(X, Y)(\mathbb{G}_m^{\times n}) \rightarrow \mathcal{A}(X', Y)(\mathbb{G}_m^{\times n})$$

as follows:  $f_n^*(\Phi, \theta_1, \dots, \theta_n) = (f^*(\Phi), f^*(\theta_1), \dots, f^*(\theta_n))$  on objects and  $f_n^*(\alpha) = f^*(\alpha)$  on morphisms.

Using Notation 4.13 for a morphism  $g : Y \rightarrow Y'$  in  $Sm/k$ , define an additive functor

$$g_{*,n} : \mathcal{A}(X, Y)(\mathbb{G}_m^{\times n}) \rightarrow \mathcal{A}(X, Y')(\mathbb{G}_m^{\times n})$$

as follows:  $g_{*,n}(\Phi, \theta_1, \dots, \theta_n) = (g_*(\Phi), g_*(\theta_1), \dots, g_*(\theta_n))$  on objects and  $g_{*,n}(\alpha) = g_*(\alpha)$  on morphisms.

**Definition 4.26.** Given  $X \in Sm/k, Y \in Sm/k$  and  $n > 0$ , we define an additive functor

$$\rho_{X,Y,n} : \mathcal{A}(X, Y \times \mathbb{G}_m^{\times n}) \rightarrow \mathcal{A}(X, Y)(\mathbb{G}_m^{\times n})$$

by using the functor  $(pr_Y)_* : \mathcal{A}(X, Y \times \mathbb{G}_m^{\times n}) \rightarrow \mathcal{A}(X, Y)$  from Notation 4.13 as follows. Given an object  $\Phi \in \mathcal{A}(X, Y \times \mathbb{G}_m^{\times n})$  and its representative

$$(n, Z, \varphi : p_{X,*}(\mathcal{O}_Z) \rightarrow M_n(\mathcal{O}_X)),$$

we have  $n$  automorphisms  $[t_i]$ 's of  $\Phi$  of the form  $m \mapsto \varphi(t_i|_Z)m$ , where each  $t_i = p_i^*(t) \in \Gamma(X \times Y \times \mathbb{G}_m^{\times n})$  and  $p_i : X \times Y \times \mathbb{G}_m^{\times n} \rightarrow \mathbb{G}_m$  is the projection. One sets

$$\rho_{X,Y,n}(\Phi) = ((pr_Y)_*(\Phi), (pr_Y)_*([t_1]), \dots, (pr_Y)_*([t_n]))$$

on objects and  $\rho_{X,Y,n}(\Phi)(\alpha) = (pr_Y)_*(\alpha)$  on morphisms.

The following lemma is a straightforward consequence of Corollary 4.14.



**Lemma 4.27.** *The bivariant additive category*

$$\mathcal{A} : (Sm/k)^{op} \times Sm/k \rightarrow AddCats, \quad (X, Y) \mapsto \mathcal{A}(X, Y),$$

*satisfies the following property:*

(Aut) *for every  $X \in Sm/k, Y \in Sm/k$  and  $n > 0$ , the functors  $\rho_{X,Y,n}$  meet the following two conditions:*

- (a) *for any  $f : X' \rightarrow X$  in  $Sm/k$  and  $n > 0$  one has  $f_n^* \circ \rho_{X,Y,n} = \rho_{X',Y,n} \circ f^*$ , where  $f^* : \mathcal{A}(X, Y \times \mathbb{G}_m^{\times n}) \rightarrow \mathcal{A}(X', Y \times \mathbb{G}_m^{\times n})$  is defined in Notation 4.13;*
- (b) *using Notation 4.13, for any  $g : Y \rightarrow Y'$  in  $Sm/k$  and  $n > 0$  one has*

$$g_{*,n} \circ \rho_{X,Y,n} = \rho_{X,Y',n} \circ (g \times id_n)_*,$$

*where  $id_n$  is the identity morphism of  $\mathbb{G}_m^{\times n}$ .*

The following proposition is true as well.

**Proposition 4.28.** *For every  $X \in Sm/k, Y \in Sm/k$  and  $n > 0$  the additive functor*

$$\rho_{X,Y,n} : \mathcal{A}(X, Y \times \mathbb{G}_m^{\times n}) \rightarrow \mathcal{A}(X, Y)(\mathbb{G}_m^{\times n})$$

*is a category isomorphism (it is not just an equivalence of categories).*

## 5. COMPARING $\mathcal{A}(X, Y)$ WITH $\tilde{\mathcal{P}}(X, Y)$

Let  $X, Y$  be two  $k$ -schemes of finite type over the base field  $k$ . We denote by  $\mathcal{P}(X, Y)$  the category of coherent  $\mathcal{O}_{X \times Y}$ -modules  $P_{X,Y}$  such that  $\text{Supp}(P_{X,Y})$  is finite over  $X$  and the coherent  $\mathcal{O}_X$ -module  $(p_X)_*(P_{X,Y})$  is locally free. A disadvantage of the category  $\mathcal{P}(X, Y)$  is that whenever we have two maps  $f : X \rightarrow X'$  and  $g : X' \rightarrow X''$  then the functor  $(g \circ f)^*$  agrees with  $f^* \circ g^*$  only up to a canonical isomorphism. To fix the problem, we replace  $\mathcal{P}(X, Y)$  by the equivalent additive category of big bimodules  $\tilde{\mathcal{P}}(X, Y)$  which is functorial in both arguments. This is done in Appendix.

In this section for any  $X \in Sm/k$  and  $Y \in AffSm/k$  a canonical functor

$$F_{X,Y} : \mathcal{A}(X, Y) \rightarrow \tilde{\mathcal{P}}(X, Y)$$

is constructed. Logically, one should now read Appendix about big bimodules, and then return to this section.

As an application, we obtain canonical isomorphisms over a perfect field  $k$

$$K_i(X) \cong DK_-^{eff}(k)(M_{\mathbb{K}}(X)[i], M_{\mathbb{K}}(pt)), \quad X \in Sm/k, i \in \mathbb{Z}, pt = \text{Spec } k,$$

where  $K(X)$  is the algebraic  $K$ -theory spectrum defined as the Waldhausen symmetric  $K$ -theory spectrum  $K(\tilde{\mathcal{P}}(X, pt))$  and  $DK_-^{eff}(k)$  is the triangulated category of  $K$ -motives (see Definition 4.23).

Let  $X, Y \in Sm/k$  and assume that  $Y$  is affine. Let  $\mathcal{A}(X, Y)$  be the additive category defined in Section 4 and let  $\tilde{\mathcal{P}}(X, Y)$  be the additive category of big bimodules defined in Appendix. If  $f : X' \rightarrow X$  is a morphism in  $Sm/k$ , then there is an additive functor  $f^* : \mathcal{A}(X, Y) \rightarrow \mathcal{A}(X', Y)$  defined in Notation 4.13. By Corollary 4.14 the assignments  $X \mapsto \mathcal{A}(X, Y)$  and  $f \mapsto f^*$  yield a presheaf of small additive categories on  $Sm/k$ . By Lemma A.1 the assignments  $X \mapsto \tilde{\mathcal{P}}(X, Y)$  and  $f \mapsto (f^* : \tilde{\mathcal{P}}(X, Y) \rightarrow \tilde{\mathcal{P}}(X', Y))$  yield another presheaf of small additive categories on  $Sm/k$ .

The main goal of this section is to prove the following

**Theorem 5.1.** *Let  $Y$  be an affine  $k$ -smooth variety. Then there is a morphism*

$$F : \mathcal{A}(-, Y) \rightarrow \tilde{\mathcal{P}}(-, Y)$$

*of presheaves of additive categories on  $\text{Sm}/k$  such that for any  $k$ -smooth affine  $X$  the functor  $F_{X,Y} : \mathcal{A}(X, Y) \rightarrow \tilde{\mathcal{P}}(X, Y)$  is an equivalence of categories.*

We postpone the proof but first construct a functor

$$F_{X,Y} : \mathcal{A}(X, Y) \rightarrow \tilde{\mathcal{P}}(X, Y)$$

which is a category equivalence whenever  $X$  is affine. We shall do this in several steps. Let  $\Phi \in \mathcal{A}(X, Y)$  be an object. It is represented by a triple

$$(n, Z, \varphi : p_{X,*}(\mathcal{O}_Z) \rightarrow M_n(\mathcal{O}_X)),$$

where  $n$  is a nonnegative integer,  $Z \in \text{SubSch}(X \times Y/Y)$  and  $\varphi$  is a non-unital homomorphism of sheaves of  $\mathcal{O}_X$ -algebras. Thus one can consider the composite of non-unital  $k$ -algebra homomorphisms

$$\Phi_X : k[Y] \rightarrow k[X \times Y] \rightarrow k[Z] \xrightarrow{\varphi} M_n(k[X]).$$

Clearly, it does not depend on the choice of a triple representing the object  $\Phi$ .

Let  $\text{Sch}/X$  be the category of  $X$ -schemes of finite type. For an  $X$ -scheme  $f : U \rightarrow X$  in  $\text{Sch}/X$  set

$$\Phi_U := M_n(f^*) \circ \Phi_X : k[Y] \rightarrow M_n(k[U]).$$

Note that  $\Phi_U$  depends not only on  $U$  itself but rather on the  $X$ -scheme  $U$ . The assignment  $U/X \mapsto \Phi_U$  defines a morphism of presheaves of non-unital  $k$ -algebras  $(U/X \mapsto k[Y]) \rightarrow (U/X \mapsto M_n(k[U]))$ .

One has a compatible family of projectors given by  $U/X \mapsto p_U^\Phi = \Phi_U(1) \in M_n(k[U])$ . Set  $P_U^\Phi = \text{Im}(p_U^\Phi) \subset k[U]^n$ . Then the assignment

$$(11) \quad U \mapsto P_U^\Phi$$

is a presheaf of  $U/X \mapsto k[U]$ -modules. The presheaf of  $U/X \mapsto k[U]$ -modules  $U/X \mapsto P_U^\Phi$  has, moreover, a  $k[Y]$ -module structure. Namely, for each  $U/X \in \text{Sch}/X$  the  $k$ -algebra  $k[Y]$  acts on  $P_U^\Phi$  by means of the non-unital  $k$ -algebra homomorphism  $\Phi_U : k[Y] \rightarrow M_n(k[U])$ . Thus the assignment  $U \mapsto P_U^\Phi$  is a presheaf of  $U/X \mapsto k[U] \otimes_k k[Y]$ -modules.

For each  $U/X \in \text{Sch}/X$  and each point  $u \in U$  set

$$p_{U,u}^\Phi := \text{colim}_{u \in V \subset U} p_V^\Phi \in M_n(\mathcal{O}_{U,u}), \quad P_{U,u}^\Phi := \text{Im}(p_{U,u}^\Phi) \subset \mathcal{O}_{U,u}^n.$$

The stalk of the  $U/X \mapsto k[U] \otimes_k k[Y]$ -module  $P_U^\Phi$  at  $u \in U$  is  $P_{U,u}^\Phi$ . So  $P_{U,u}^\Phi$  is an  $\mathcal{O}_{U,u} \otimes_k k[Y]$ -module.

**Definition 5.2.** Let  $U/X \in \text{Sch}/X$  and  $q \in U \times Y$  be a point. Let  $u = pr_U(q) \in U$  be its image in  $U$ . Set

$$\mathcal{P}_{U,q}^\Phi := \left\{ \frac{m}{g} \mid m \in P_{U,u}^\Phi, g \in \mathcal{O}_{U,u} \otimes_k k[Y] \text{ such that } g(q) \neq 0 \right\} / \sim,$$

where “ $\sim$ ” is the standard equivalence relation for fractions. Clearly,  $\mathcal{P}_{U,q}^\Phi$  is an  $\mathcal{O}_{U \times Y, q}$ -module.

Now define a Zarisky sheaf  $\mathcal{P}_U^\Phi$  of  $\mathcal{O}_{U \times Y}$ -modules on  $U \times Y$  as follows. Its sections on an open set  $W \subset U \times Y$  are a compatible family of elements  $\{n_q \in \mathcal{P}_{U,q}^\Phi\}_{q \in W}$ . More precisely, we give the following

**Definition 5.3.** Set  $\mathcal{P}_U^\Phi(W)$  to consist of the tuples  $(n_q) \in \prod_{q \in W} \mathcal{P}_{U,q}^\Phi$  such that there is an affine cover  $U = \cup U_i$  and for any  $i$  there is an affine cover of the form  $(W \cap U_i \times Y) = \cup (U_i \times Y)_{g_{ij}}$  with  $g_{ij} \in k[U_i \times Y]$  and there are elements  $n_{ij} \in (P_{U_i}^\Phi)_{g_{ij}}$  such that for any  $i$  and any  $i_j$  and any point  $q \in (U_i \times Y)_{g_{ij}}$  one has  $n_{ij} = n_q \in \mathcal{P}_{U,q}^\Phi$ . Here  $(U_i \times Y)_{g_{ij}}$  stands for the principal open set associated with  $g_{ij}$ .

Clearly, the assignment  $W \mapsto \mathcal{P}_U^\Phi(W)$  is a Zarisky sheaf of  $\mathcal{O}_{U \times Y}$ -modules on  $U \times Y$ . The  $\mathcal{O}_{U \times Y}$ -module structure on this sheaf is given as follows: for  $f \in k[W]$  and  $(n_q) \in \mathcal{P}_U^\Phi(W)$  set  $f \cdot (n_q) = (f \cdot n_q)$ .

Next, for any morphism  $f : V \rightarrow U$  of objects in  $Sch/X$  construct a sheaf morphism

$$\sigma_f : \mathcal{P}_U^\Phi \rightarrow F_*(\mathcal{P}_V^\Phi),$$

where  $F = f \times \text{id} : V \times Y \rightarrow U \times Y$ . Given a point  $v \in V$  and its image  $u \in U$ , set  $F_v^* = p_{V,v}^\Phi \circ f^* \circ i_{U,u}^\Phi : P_{U,u}^\Phi \rightarrow P_{V,v}^\Phi$ , where  $i_{U,u}^\Phi : P_{U,u}^\Phi \hookrightarrow \mathcal{O}_{U,u}^n$  is the inclusion.

For any point  $r \in V \times Y$  and its image  $s = F(r) \in U \times Y$  set  $v = pr_V(r)$  and  $u = pr_U(s)$ . Clearly,  $f(v) = u$ . The  $k$ -algebra homomorphism  $\mathcal{O}_{U \times Y,s} \rightarrow \mathcal{O}_{V \times Y,r}$  makes  $\mathcal{P}_{V,r}^\Phi$  an  $\mathcal{O}_{U \times Y,s}$ -module. There is a unique homomorphism  $F_r^* : \mathcal{P}_{U,s}^\Phi \rightarrow \mathcal{P}_{V,r}^\Phi$  of  $\mathcal{O}_{U \times Y,s}$ -modules making the diagram commutative

$$\begin{array}{ccc} P_{U,u}^\Phi & \longrightarrow & \mathcal{P}_{U,s}^\Phi \\ F_v^* \downarrow & & \downarrow F_r^* \\ P_{V,v}^\Phi & \longrightarrow & \mathcal{P}_{V,r}^\Phi \end{array}$$

Let  $W \subset U \times Y$  be an open subset. By definition,

$$\mathcal{P}_U^\Phi(W) = \{(n_s) \in \prod_{s \in W} \mathcal{P}_{U,s}^\Phi \mid n_s \text{ are locally compatible}\}$$

and

$$F_*(\mathcal{P}_V^\Phi)(W) = \mathcal{P}_V^\Phi(F^{-1}(W)) = \{(n_r) \in \prod_{r \in F^{-1}(W)} \mathcal{P}_{V,r}^\Phi \mid n_r \text{ are locally compatible}\}.$$

Define  $F_W^* : \mathcal{P}_U^\Phi(W) \rightarrow F_*(\mathcal{P}_V^\Phi)(W)$  as follows. Given a section  $(n_s \in \mathcal{P}_{U,s}^\Phi)_{s \in W}$  of  $\mathcal{P}_U^\Phi$  over  $W$ , set

$$F_W^*((n_s \in \mathcal{P}_{U,s}^\Phi)_{s \in W}) := ((F_r^*(n_s)_{f(r)=s}))_{s \in W}.$$

It is straightforward to check that the assignment  $W \mapsto F_W^*$  defines an  $\mathcal{O}_{U \times Y}$ -sheaf morphism

$$\sigma_f : \mathcal{P}_U^\Phi \rightarrow F_*(\mathcal{P}_V^\Phi).$$

Moreover, for a pair of morphisms  $g : U_3 \rightarrow U_2$  and  $f : U_2 \rightarrow U_1$  in  $Sch/X$  one has

$$\sigma_{f \circ g} = (f \times \text{id}_Y)_*(\sigma_g) \circ \sigma_f : \mathcal{P}_{U_1}^\Phi \rightarrow (F \circ G)_*(\mathcal{P}_{U_3}^\Phi) = F_*(G_*(\mathcal{P}_{U_3}^\Phi)).$$

**Lemma 5.4.** The data  $U/X \mapsto \mathcal{P}_U^\Phi$  and  $(f : V \rightarrow U) \mapsto (\sigma_f : \mathcal{P}_U^\Phi \rightarrow F_*(\mathcal{P}_V^\Phi))$  defined above determine an object of the category  $\tilde{\mathcal{P}}(X, Y)$ . We shall denote this object by  $F_{X,Y}(\Phi)$ .

Now define the functor  $F_{X,Y} : \mathcal{A}(X, Y) \rightarrow \tilde{\mathcal{P}}(X, Y)$  on morphisms. Let  $\alpha : \Phi \rightarrow \Psi$  be a morphism in  $\mathcal{A}(X, Y)$ . The morphism  $\alpha$  is a Zarisky sheaf morphism

$$(U/X \mapsto P_U^\Phi) \rightarrow (U/X \mapsto P_U^\Psi)$$

on small Zarisky site  $X_{Zar}$  respecting the  $k[Y]$ -module structure on both sides. We write  $\alpha_U : P_U^\Phi \rightarrow P_U^\Psi$  for the value of  $\alpha$  at  $U$ . For any point  $x \in X$  the Zarisky sheaf morphism  $\alpha$  induces a morphism of stalks

$$\alpha_x : P_{X,x}^\Phi \rightarrow P_{X,x}^\Psi.$$

Finally, for any point  $q \in X \times Y$  and its image  $x = p_X(q) \in X$  one has a homomorphism

$$\alpha_q : \mathcal{P}_{X,q}^\Phi \rightarrow \mathcal{P}_{X,q}^\Psi$$

given by  $\alpha_q(\frac{m}{g}) = \frac{\alpha_x(m)}{g}$  for any  $m \in P_{X,x}^\Phi$  and any  $g \in \mathcal{O}_{X,x} \otimes_k k[Y]$  with  $g(q) \neq 0$ .

**Definition 5.5.** Define a morphism  $F_{X,Y}(\alpha) : F_{X,Y}(\Phi) \rightarrow F_{X,Y}(\Psi)$  as follows. Given a Zarisky open subset  $W \subset X \times Y$  and a section  $s = (n_q) \in \mathcal{P}_X^\Phi(W)$ , set  $\alpha_W(s) = (\alpha_q(n_q))$ . Clearly, the family  $(\alpha_q(n_q))$  is an element of  $\mathcal{P}_X^\Psi(W)$ . Moreover,  $\alpha_W$  is a homomorphism and the assignment  $W \mapsto \alpha_W$  is a morphism in  $\tilde{\mathcal{P}}(X, Y)$ . We shall write  $F_{X,Y}(\alpha)$  for this morphism in  $\tilde{\mathcal{P}}(X, Y)$ .

**Lemma 5.6.** *The assignments  $\Phi \mapsto F_{X,Y}(\Phi)$  and  $\alpha \mapsto F_{X,Y}(\alpha)$  determine an additive functor  $F_{X,Y} : \mathcal{A}(X, Y) \rightarrow \tilde{\mathcal{P}}(X, Y)$ . Moreover, for a given affine  $k$ -smooth variety  $Y$  the assignment  $X \mapsto F_{X,Y}$  determines a morphism of presheaves of additive categories.*

Lemma 5.6 implies that in order to prove Theorem 5.1, it remains to check that for affine  $X, Y \in \text{AffSm}/k$  the functor  $F_{X,Y}$  is a category equivalence. Firstly describe a plan of the proof. Given  $X, Y \in \text{AffSm}/k$  we shall construct a square of additive categories and additive functors

$$(12) \quad \begin{array}{ccc} \mathcal{A}(X, Y) & \xrightarrow{F_{X,Y}} & \tilde{\mathcal{P}}(X, Y) \\ \Gamma \downarrow & & \downarrow R \\ A(X, Y) & \xrightarrow{a_{X,Y}} & \mathcal{P}(X, Y) \end{array}$$

which commutes up to an isomorphism of additive functors. We shall prove that the functors  $\Gamma$ ,  $a_{X,Y}$  and  $R$  are equivalences of categories. As a consequence, the functor  $F_{X,Y}$  will be an equivalence of categories.

**Definition 5.7.** For affine schemes  $X, Y \in \text{AffSm}/k$  define a category  $A(X, Y)$  as follows. Objects of  $A(X, Y)$  are the pairs  $(n, \varphi)$ , where  $n \geq 0$  and  $\varphi : k[Y] \rightarrow M_n(k[X])$  is a non-unital  $k$ -algebra homomorphism. The homomorphism  $\varphi$  defines a projector  $\varphi(1) \in M_n(k[X])$ . The projector  $\varphi(1)$  defines a projective  $k[X]$ -module  $\text{Im}(\varphi(1) : k[X]^n \rightarrow k[X]^n)$ . This  $k[X]$ -module has also a  $k[Y]$ -module structure which is given by the non-unital homomorphism  $\varphi$ . Namely,  $m\varphi := \varphi(f)(m)$ . Thus  $\text{Im}(\varphi(1))$  is a  $k[X \times Y]$ -module. Set

$$\text{Mor}_{A(X,Y)}((n_1, \varphi_1), (n_2, \varphi_2)) = \text{Hom}_{k[X \times Y]}(\text{Im}(\varphi_1(1)), \text{Im}(\varphi_2(1))).$$

**Definition 5.8.** Given affine schemes  $X, Y \in \text{AffSm}/k$ , define a functor

$$\Gamma : \mathcal{A}(X, Y) \rightarrow A(X, Y)$$

as follows. Given an object  $\Psi \in \mathcal{A}(X, Y)$ , choose its representative  $(n, Z, \psi : p_{X,*}(\mathcal{O}_Z) \rightarrow M_n(\mathcal{O}_X))$ . This representative gives rise to a pair

$$\Gamma(\Psi) := (n, \varphi : k[Y] \rightarrow k[X \times Y] \rightarrow \Gamma(Z, \mathcal{O}_Z) \xrightarrow{\psi} M_n(k[X])),$$

which is an object of  $A(X, Y)$ . Clearly, this pair does not depend on the choice of a representative. One has an equality  $\Gamma(X, P(\psi)) = P_X^\Psi$ , where  $P(\psi)$  is defined in Definition 4.5 and  $P_X^\Psi$  is given

by (11). If  $\alpha : \Psi_1 \rightarrow \Psi_2$  is a morphism in  $\mathcal{A}(X, Y)$ , then equalizing the supports of  $\Psi_1$  and  $\Psi_2$  and taking the global sections on  $X$ , we get an isomorphism

$$\begin{aligned} \text{Mor}_{\mathcal{A}(X, Y)}(\Psi_1, \Psi_2) &= \text{Hom}_{p_{X,*}(\mathcal{O}_Z)}(P(\Psi_1), P(\Psi_2)) \xrightarrow{\Gamma(\alpha)} \\ &\xrightarrow{\Gamma(\alpha)} \text{Hom}_{k[X \times Y]}(P_X^{\Psi_1}, P_X^{\Psi_2}) = \text{Hom}_{A(X, Y)}(\Gamma(\Psi_1), \Gamma(\Psi_2)). \end{aligned}$$

This completes the definition of the functor  $\Gamma$ .

**Lemma 5.9.** *The functor  $\Gamma : \mathcal{A}(X, Y) \rightarrow A(X, Y)$  is an equivalence of additive categories.*

*Proof.* Define a functor  $a : A(X, Y) \rightarrow \mathcal{A}(X, Y)$  on objects as follows. An object  $(n, \varphi)$  in  $A(X, Y)$  defines a projector  $\varphi(1) \in M_n(k[X])$ . The image  $\text{Im}(\varphi(1))$  in  $k[X]^n$  has a  $k[Y]$ -module structure given by the non-unital homomorphism  $\varphi$ . In this way  $\text{Im}(\varphi(1))$  is a  $k[X \times Y]$ -module. Let  $A \subset X \times Y$  be the support of  $\text{Im}(\varphi(1))$ . Using Notation 4.1, it is easy to see that  $A \in \text{Supp}(X \times Y/Y)$ . Thus there exists an integer  $m \geq 0$  such that  $I_A^m \cdot \text{Im}(\varphi(1)) = (0)$ . The latter means that  $\text{Im}(\varphi(1))$  is a  $k[X \times Y]/I_A^m$ -module, and therefore the non-unital  $k$ -algebra homomorphism  $\varphi$  can be presented in the form

$$k[X \times Y] \rightarrow k[X \times Y]/I_A^m \xrightarrow{\bar{\varphi}_{A,m}} M_n(k[X])$$

for a unique  $\bar{\varphi}_{A,m}$ . Let  $Z = \text{Spec}(k[X \times Y]/I_A^m)$  and let  $(\bar{\varphi}_{A,m})^\sim : p_{X,*}(\mathcal{O}_Z) \rightarrow M_n(\mathcal{O}_X)$  be the sheaf homomorphism associated to  $\bar{\varphi}_{A,m}$ . Set

$$a(n, \varphi) := \text{the equivalence class of the triple } (n, Z, (\bar{\varphi}_{A,m})^\sim).$$

Clearly, this equivalence class remains unchanged when enlarging  $A$  in  $\text{Supp}(X \times Y/Y)$  and the integer  $m$ .

Define the functor  $a : A(X, Y) \rightarrow \mathcal{A}(X, Y)$  on morphisms as follows. Let  $\alpha : (n_1, \varphi_1) \rightarrow (n_2, \varphi_2)$  be a morphism in  $A(X, Y)$ . Let  $A_i$  be the support of the  $k[X \times Y]$ -module  $\text{Im}(\varphi_i(1))$  and let  $m_i$  be an integer such that  $I_{A_i}^{m_i} \cdot \text{Im}(\varphi_i(1)) = (0)$ . Enlarging  $A_1$  and  $A_2$  in  $\text{Supp}(X \times Y/Y)$  if necessary, we may assume that  $A_1 = A = A_2$ . Enlarging  $m_1$  and  $m_2$ , we may as well assume that  $m_1 = m = m_2$ . Therefore we may assume that  $Z_1 = Z = Z_2$ . Now applying the functor from  $k[X]$ -modules to  $\mathcal{O}_X$ -modules, we get a homomorphism

$$\begin{aligned} \text{Hom}_{A(X, Y)}((n_1, \varphi_1), (n_2, \varphi_2)) &= \text{Hom}_{k[X \times Y]}(\text{Im}(\varphi_1(1)), \text{Im}(\varphi_2(1))) = \\ &= \text{Hom}_{k[X \times Y]/I_A^m}(\text{Im}(\bar{\varphi}_{1,A,m}(1)), \text{Im}(\bar{\varphi}_{2,A,m}(1))) \rightarrow \text{Hom}_{p_{X,*}(\mathcal{O}_Z)}(\text{Im}(\bar{\varphi}_1(1))^\sim, \text{Im}(\bar{\varphi}_2(1))^\sim). \end{aligned}$$

Set  $a(\alpha)$  to be the image of  $\alpha$  under this homomorphism. The definition of the functor  $a$  is completed.

It is straightforward to check that the functors  $\Gamma$  and  $a$  are mutually inverse equivalences of additive categories.  $\square$

**Definition 5.10.** Define a functor  $a_{X,Y} : A(X, Y) \rightarrow \mathcal{P}(X, Y)$  as follows. It takes an object  $(n, \varphi)$  to the  $\mathcal{O}_{X \times Y}$ -module sheaf  $\text{Im}(\varphi(1))^\sim$  associated with the  $k[X \times Y]$ -module  $\text{Im}(\varphi(1))$  described in Definition 5.7. On morphisms it is defined by the isomorphism

$$\begin{aligned} \text{Hom}_{A(X, Y)}((n_1, \varphi_1), (n_2, \varphi_2)) &= \text{Hom}_{k[X \times Y]}(\text{Im}(\varphi_1(1)), \text{Im}(\varphi_2(1))) \cong \\ &\cong \text{Hom}_{\mathcal{O}_{X \times Y}}(\text{Im}(\varphi_1(1))^\sim, \text{Im}(\varphi_2(1))^\sim) = \text{Hom}_{\mathcal{P}(X, Y)}(a_{X,Y}(n_1, \varphi_1), a_{X,Y}(n_2, \varphi_2)). \end{aligned}$$

*Proof of Theorem 5.1.* Consider the following square of functors

$$\begin{array}{ccc} \mathcal{A}(X, Y) & \xrightarrow{F_{X,Y}} & \tilde{\mathcal{P}}(X, Y) \\ \Gamma \downarrow & & \downarrow R \\ A(X, Y) & \xrightarrow{a_{X,Y}} & \mathcal{P}(X, Y) \end{array}$$

where  $R$  takes a big bimodule  $P \in \tilde{\mathcal{P}}(X, Y)$  to the  $\mathcal{O}_{X \times Y}$ -module  $P_{X,Y} \in \mathcal{P}(X, Y)$  and a morphism  $\alpha : P \rightarrow Q$  of big bimodules to the morphism  $\alpha_{X,Y} : P_{X,Y} \rightarrow Q_{X,Y}$  of  $\mathcal{O}_{X \times Y}$ -modules. We claim that this diagram commutes up to an isomorphism of functors. Since the functors  $\Gamma$ ,  $a_{X,Y}$ ,  $R$  are equivalences of categories, the functor  $F_{X,Y}$  is a category equivalence, too. To complete the proof, it remains to construct a functor isomorphism  $a_{X,Y} \circ \Gamma \rightarrow R \circ F_{X,Y}$ .

Let  $\Psi \in \mathcal{A}(X, Y)$  be an object and let  $(n, Z, \psi : p_{X,*}(\mathcal{O}_Z) \rightarrow M_n(\mathcal{O}_X))$  be a triple representing  $\Psi$  (see Definition 4.5). Then  $\Gamma(\Psi) = (n, \varphi : k[Y] \rightarrow k[X \times Y] \rightarrow \Gamma(Z, \mathcal{O}_Z) \xrightarrow{\psi} M_n(k[X]))$  as described in Definition 5.8. Let  $\text{Im}(\varphi(1))$  be the  $k[X \times Y]$ -module described in Definition 5.7. Then  $a_{X,Y}(\Gamma(\Psi))$  is the  $\mathcal{O}_{X \times Y}$ -module sheaf  $\text{Im}(\varphi(1))^\sim$  associated with the  $k[X \times Y]$ -module  $\text{Im}(\varphi(1))$ . On the other hand, following Definition 5.3 and the description of  $R$ , one has  $R(F_{X,Y}(\Psi)) = \mathcal{P}_X^\Psi$ . We need to construct an isomorphism  $\theta_\Psi : \text{Im}(\varphi(1))^\sim \xrightarrow{\cong} \mathcal{P}_X^\Psi$ , natural in  $\Psi$ , of  $\mathcal{O}_{X \times Y}$ -modules. Giving such a morphism  $\theta_\Psi$  is the same as giving a  $k[X \times Y]$ -homomorphism

$$\Theta_\Psi : \text{Im}(\varphi(1)) \rightarrow \Gamma(X \times Y, \mathcal{P}_X^\Psi).$$

Moreover,  $\theta_\Psi$  is an isomorphism whenever so is  $\Theta_\Psi$ . A section of  $\mathcal{P}_X^\Psi$  on  $X \times Y$  is a compatible family of elements  $(n_q \in \mathcal{P}_{X,q}^\Psi)_{q \in X \times Y}$  (see Definitions 5.3 and 5.2). For  $s \in \text{Im}(\varphi(1))$ , set

$$\Theta_\Psi(s) = \left( \frac{s_{x(q)}}{1} \right) \in \prod_{q \in X \times Y} \mathcal{P}_{X,q}^\Psi,$$

where  $x(q) = p_X(q) \in X$  and  $s_{x(q)} \in P_{X,x(q)}^\Psi$  is the image of  $s$  in  $P_{X,x(q)}^\Psi$  under the canonical map  $P_X^\Psi = \text{Im}(p_X^\Psi) \rightarrow \text{Im}(p_{X,x(q)}^\Psi) = P_{X,x(q)}^\Psi$  (see the discussion above Definition 5.2). Clearly,  $\Theta_\Psi(s)$  belongs to  $\Gamma(X \times Y, \mathcal{P}_X^\Psi)$ . We claim that  $\Theta_\Psi$  is an isomorphism. In fact, if  $\frac{s_{x(q)}}{1} = 0$  for all  $q \in X \times Y$  then  $s = 0$ . It follows that  $\Theta_\Psi$  is injective. If  $(n_q) \in \Gamma(X \times Y, \mathcal{P}_X^\Psi)$ , then  $(n_q) \in \prod_{q \in X \times Y} \mathcal{P}_{X,q}^\Psi$  is a compatible family of elements. It follows from Definition 5.3 that there is a global section  $s$  of the sheaf  $\text{Im}(\varphi(1))^\sim$  such that for each  $q \in X \times Y$  one has  $s_q = n_q$ . Since  $\Gamma(X \times Y, \text{Im}(\varphi(1))^\sim) = \text{Im}(\varphi(1))$  the map  $\Theta_\Psi$  is surjective. The fact that the assignment  $\Psi \mapsto \Theta_\Psi$  is a functor transformation  $a_{X,Y} \circ \Gamma \rightarrow R \circ F_{X,Y}$  is obvious. Our theorem now follows.  $\square$

Recall that the  $\mathbb{K}$ -motive  $M_{\mathbb{K}}(X)$  of a  $k$ -smooth scheme  $X$  is the  $\mathbb{K}$ -module  $C_*(\mathbb{K}(-, X))$  (see Definition 3.3). To conclude the section, we give the following computational application of Theorem 5.1.

**Theorem 5.11.** *Let  $k$  be a perfect field and let  $X$  be any scheme in  $\text{Sm}/k$ . Then for every integer  $i \in \mathbb{Z}$  there is a natural isomorphism of abelian groups*

$$K_i(X) \cong DK_-^{eff}(k)(M_{\mathbb{K}}(X)[i], M_{\mathbb{K}}(pt)),$$

where  $K(X)$  is algebraic  $K$ -theory of  $X$ .

*Proof.* By Theorem 4.22  $\mathbb{K}$  is a strict  $V$ -spectral category. By (3) one has a canonical isomorphism for every integer  $i$

$$SH_{S^1}^{\text{nis}}(k)(X[i], \mathbb{K}(-, pt)) \cong SH_{S^1}^{\text{nis}}\mathbb{K}(\mathbb{K}(-, X)[i], \mathbb{K}(-, pt)).$$

Let

$$K : Sm/k \rightarrow Sp^\Sigma, \quad X \mapsto K(X) = K(\tilde{\mathcal{P}}(X, pt))$$

be the algebraic  $K$ -theory presheaf of symmetric spectra. It follows from Theorem 5.1 that the natural map in  $Pre^\Sigma(Sm/k)$

$$F : \mathbb{K}(-, pt) \rightarrow K,$$

induced by the additive functors  $F_{X, pt} : \mathcal{A}(X, pt) \rightarrow \tilde{\mathcal{P}}(X, pt)$ ,  $X \in Sm/k$ , is a Nisnevich local weak equivalence.

Using Thomason's theorem [13] stating that algebraic  $K$ -theory satisfies Nisnevich descent, we obtain isomorphisms

$$K_i(X) \cong SH_{S^1}^{\text{nis}}(k)(X[i], K) \cong SH_{S^1}^{\text{nis}}\mathbb{K}(\mathbb{K}(-, X)[i], \mathbb{K}(-, pt)), \quad i \in \mathbb{Z}.$$

Consider a commutative diagram in  $Pre^\Sigma(Sm/k)$

$$\begin{array}{ccccc} K(\mathcal{A}(-, pt)) & \longrightarrow & K(\mathcal{A}(-, pt))_f & \longrightarrow & |\underline{\text{Hom}}(\Delta^\cdot, K(\mathcal{A}(-, pt))_f)| \\ F \downarrow & & \delta \downarrow & & \downarrow \gamma \\ K & \xrightarrow{\alpha} & K_f & \xrightarrow{\beta} & |\underline{\text{Hom}}(\Delta^\cdot, K_f)| \end{array}$$

Here the lower  $f$ -symbol refers to a fibrant replacement functor in the Nisnevich local model structure on  $Pre^\Sigma(Sm/k)$ . Theorem 5.1 implies  $F$  is a Nisnevich local weak equivalence. By [13]  $K(-)$  is Nisnevich excisive, and hence  $\alpha$  is a stable weak equivalence. Since  $K(-)$  is homotopy invariant, then  $\beta$  is a stable weak equivalence. It follows that  $\delta, \gamma$  are stable weak equivalences. Therefore the composition of the upper horizontal maps is a Nisnevich local weak equivalence. Thus the canonical map

$$\mathbb{K}(-, pt) \rightarrow M_{\mathbb{K}}(pt)$$

is a Nisnevich local weak equivalence. One has an isomorphism

$$K_i(X) \cong SH_{S^1}^{\text{nis}}\mathbb{K}(\mathbb{K}(-, X)[i], M_{\mathbb{K}}(pt)), \quad i \in \mathbb{Z}.$$

Since  $\mathbb{K}(-, X)[i], M_{\mathbb{K}}(pt)$  are bounded below  $\mathbb{K}$ -modules, then our theorem follows from Theorem 3.5(2).  $\square$

## APPENDIX A. THE CATEGORY OF BIG BIMODULES $\tilde{\mathcal{P}}(X, Y)$

Let  $X, Y$  be two schemes of finite type over the base field  $k$ . We denote by  $\mathcal{P}(X, Y)$  the category of coherent  $\mathcal{O}_{X \times Y}$ -modules  $P_{X, Y}$  such that  $\text{Supp}(P_{X, Y})$  is finite over  $X$  and the coherent  $\mathcal{O}_X$ -module  $(p_X)_*(P_{X, Y})$  is locally free. A disadvantage of the category  $\mathcal{P}(X, Y)$  is that whenever we have two maps  $f : X \rightarrow X'$  and  $g : X' \rightarrow X''$  then the functor  $(g \circ f)^*$  agrees with  $f^* \circ g^*$  only up to a canonical isomorphism. We want to replace  $\mathcal{P}(X, Y)$  by an equivalent additive category  $\tilde{\mathcal{P}}(X, Y)$  which is functorial in both arguments.

To this end, we use the construction which is in spirit the same with Grayson's one for finitely generated projective modules [4]. Let  $X$  be a Noetherian scheme. Consider the big Zariski site  $Sch/X$  of all schemes of finite type over  $X$ . We define the *category of big bimodules*  $\tilde{\mathcal{P}}(X, Y)$  as follows.

An object of  $\tilde{\mathcal{P}}(X, Y)$  consists of the following data:

- (1) For any  $U \in \text{Sch}/X$  one has a bimodule  $P_{U,Y} \in \mathcal{P}(U,Y)$ .
- (2) For any morphism  $f : U' \rightarrow U$  in  $\text{Sch}/X$  one has a morphism  $\sigma_f : P_{U,Y} \rightarrow (f \times 1_Y)_*(P_{U',Y})$  in  $\mathcal{P}(U,Y)$  satisfying:
  - (a)  $\sigma_1 = 1$ .
  - (b) The morphism  $\tau_f : (f \times 1_Y)^*(P_{U,Y}) \rightarrow P_{U',Y}$  which is adjoint to  $\sigma_f$  must be an isomorphism in  $\mathcal{P}(U',Y)$ .
  - (c) Given a chain of maps  $U'' \xrightarrow{f_1} U' \xrightarrow{f} U$  in  $\text{Sch}/X$ , the following relation is satisfied

$$\sigma_{f \circ f_1} = (f \times 1_Y)_*(\sigma_{f_1}) \circ \sigma_f.$$

A morphism of two big bimodules  $\alpha : P \rightarrow Q$  is a morphism  $\alpha_{X,Y} : P_{X,Y} \rightarrow Q_{X,Y}$  in  $\mathcal{P}(X,Y)$ . Clearly,  $\mathcal{P}'(X,Y)$  is an additive category.

Given a map  $g : X' \rightarrow X$  of two Noetherian schemes, we define an additive functor

$$g^* : \tilde{\mathcal{P}}(X,Y) \rightarrow \tilde{\mathcal{P}}(X',Y)$$

as follows. For any  $U \in \text{Sch}/X'$  and  $P \in \mathcal{P}(X,Y)$  one sets  $g^*(P)_{U,Y} = P_{U,Y}$ , where  $U$  is regarded as an object of  $\text{Sch}/X$  by means of composition with  $g$ . In a similar way, if  $h : U' \rightarrow U$  is a map in  $\text{Sch}/X'$  then  $\sigma_h : g^*(P)_{U,Y} \rightarrow g^*(P)_{U',Y}$  equals  $\sigma_h$ . So we have defined  $g^*$  on objects. Let  $\alpha : P \rightarrow Q$  be a morphism in  $\tilde{\mathcal{P}}(X,Y)$ . By definition, it is a morphism  $\alpha_{X,Y} : P_{X,Y} \rightarrow Q_{X,Y}$  in  $\mathcal{P}(X,Y)$ . There is a commutative diagram

$$\begin{array}{ccc} (g \times 1_Y)^*(P_{X,Y}) & \xrightarrow{\tau_g} & P_{X',Y} \\ (g \times 1_Y)^*(\alpha_{X,Y}) \downarrow & & \downarrow \alpha_{X',Y} \\ (g \times 1_Y)^*(Q_{X,Y}) & \xrightarrow{\tau_g} & Q_{X',Y} \end{array}$$

where the horizontal maps are isomorphisms. Then  $g^*(\alpha) := \alpha_{X',Y}$ . The functor  $g^*$  is constructed.

**Lemma A.1.** *Let  $g_1 : X'' \rightarrow X'$  and  $g : X' \rightarrow X$  be two maps of schemes. Then  $(g \circ g_1)^* = g_1^* \circ g^*$ .*

*Proof.* This is straightforward.  $\square$

Now let us discuss functoriality in  $Y$ . For this consider a map  $h : Y \rightarrow Y'$ . We construct an additive functor

$$h_* : \tilde{\mathcal{P}}(X,Y) \rightarrow \tilde{\mathcal{P}}(X,Y')$$

in the following way. We set  $h_*(P)_{U,Y'} = (1_U \times h)_*(P_{U,Y})$  for any  $P \in \tilde{\mathcal{P}}(X,Y)$ . If  $f : U' \rightarrow U$  is a map in  $\text{Sch}/X$  then

$$(1_U \times h)_*(f \times 1_Y)_* = (f \times 1_{Y'})_*(1_{U'} \times h)_*.$$

We define  $\sigma_f$  for  $h_*(P)$  as

$$(1_U \times h)_*(\sigma_f) : (1_U \times h)_*(P_{U,Y}) \rightarrow (1_U \times h)_*(f \times 1_Y)_*(P_{U',Y}) = (f \times 1_{Y'})_*(1_{U'} \times h)_*(P_{U',Y}).$$

By definition,  $h_*$  takes a morphism  $\alpha_{X,Y}$  in  $\mathcal{P}(X,Y)$  to  $(1_X \times h)_*(\alpha_{X,Y})$ . The construction of the functor  $h_*$  is completed.

**Lemma A.2.** *Let  $h_1 : Y' \rightarrow Y''$  and  $h : Y \rightarrow Y'$  be two maps of schemes. Then  $(h_1 \circ h)_* = (h_1)_* \circ h_*$ .*

*Proof.* This is straightforward.  $\square$

We leave the reader to verify the following



**Proposition A.3.** *The natural functor*

$$R : \tilde{\mathcal{P}}(X, Y) \rightarrow \mathcal{P}(X, Y), \quad P \mapsto P_{X, Y},$$

*is an equivalence of additive categories.*

By Lemmas A.1-A.2  $\tilde{\mathcal{P}}(X, Y)$  has the desired functoriality properties in both arguments.

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DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETON PARK, SWANSEA SA2 8PP, UNITED KINGDOM

*E-mail address:* [g.garkusha@swansea.ac.uk](mailto:g.garkusha@swansea.ac.uk)

ST. PETERSBURG BRANCH OF V. A. STEKLOV MATHEMATICAL INSTITUTE, FONTANKA 27, 191023 ST. PETERSBURG, RUSSIA

ST. PETERSBURG STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS AND MECHANICS, UNIVERSITETSKY PROSPEKT, 28, 198504, PETERHOF, ST. PETERSBURG, RUSSIA

*E-mail address:* [paniniv@gmail.com](mailto:paniniv@gmail.com)